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JEL Classification: E4, E5 and E6
Keywords: fiscal policy, government debt, monetary policy and new keynesian model

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Optimal Monetary and Fiscal Policy in an Economy with Endogenous Public Debt

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This paper uses a New Keynesian framework to study the coordination of fiscal and monetary policies, in response to an inflation shock when the policymaker acts with commitment. We first show that, in the simplest New Keynesian model, fiscal policy plays no part in the optimal policy response, because of the comparative advantage which monetary policy has in the control of inflation. We then add endogenous public debt and show that the above result is no longer true. When the initial stock of debt is low, it is optimal for government spending to remain largely inactive, but when the initial stock of debt is high, government spending should play a significant stabilisation role in the first period. This finding is robust to adding endogenous capital accumulation and inflation persistence in the Phillips curve.

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1 Introduction

This paper uses a New Keynesian framework to study the optimal coordination of fiscal and monetary policy when the policymaker is able to act with commitment. Conventional

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wisdom asserts that in such a framework the economy should be managed by monetary policy alone, and that fiscal policy should be kept in the background, being used only to ensure that the public sector remains solvent. We challenge that view in this paper.[1]

We proceed in two stages. First we show that the conventional wisdom can be given a formal, microfounded justification, in the context of a simple new Keynesian model such as that provided by Clarida, Gali and Gertler (1999). Such a demonstration is necessary, because it is not easy to find the necessary result in the literature.[2] Second we show that this conventional view is incorrect in an economy with an endogenous stock of public debt, in which the stock of such debt is initially high. We carry out a number of robustness checks and show that our challenge to the conventional wisdom is strengthened in a general new Keynesian model in which there is an endogenous capital stock and inflation persistence.[3]

Our challenge has nothing to do with the view that fiscal policy should be used when monetary policy has reached the zero bound, or when the country is a member of a monetary union. We discuss these points in the conclusion to the paper.

1.1 Summary

The conventional view asserts that a policymaker should not use both monetary and fiscal policies to stabilise inflation, even when this is possible. This claim contrasts with the (seemingly reasonable) intuition that, in the face of a positive inflation shock, it would improve welfare if government spending were reduced, alongside the fall in consumption that follows a policy-induced increase in the nominal interest rate. Such intuition relies on the idea that, because a reduction in consumption causes a welfare loss since it interferes with consumption smoothing, it might be possible to raise welfare by inducing some reduction in government spending, so that there can be a smaller fall in consumption. The conventional view argues that such intuition is incorrect.

In Section 2 we provide a formal proof of the conventional view that, within the framework of a simple standard New Keynesian model without public debt accumulation, in

[1]The findings obtainable when the policymaker acts with discretion are mentioned briefly at the relevant places in the paper; full results are obtainable from the authors on request. Fiscal activism is *more* likely in these circumstances.

[2]See Subsection 1.2 below for a review of that literature.

[3]Examples of such models include Christiano, Eichenbaum and Evans (2005), Smets and Wouters (2007) and Christiano, Trabandt and Walentin (2011).

which government expenditure is financed by lump sum taxes, it is optimal not to use fiscal policy in the stabilisation of inflation. Our result depends on the fact that consumption has a 'comparative advantage' in controlling inflation; *i.e.* monetary policy brings about a larger reduction in inflation than fiscal policy per unit of demand reduction. The comparative advantage of monetary policy comes about for the following reason. A rise in the interest rate leads to a fall in consumption which not only reduces demand, but also makes consumers more willing to increase the supply of labour. Both of these things moderate inflation. By contrast, a fall in government spending only reduces demand, and so influences inflation in only one way. In any given period, the policymaker can achieve a certain level of inflation either by raising the interest rate so as to reduce consumption or by cutting government spending. But, due to the extra effect of the interest rate on labour supply, it is always better to stabilise inflation by relying on an increase in the interest rate, without making any use of fiscal policy.

The conventional position is not normally justified in this way, but in two different, but inadequate, ways. It is asserted that fiscal policy only works with a lag whereas monetary policy does not (Blinder and Solow, 1973), and that fiscal policy is more likely to be influenced by political pressures than a monetary policy which is made by an independent central bank (Alesina and Perotti, 1995; Alesina, Perotti and Tavares, 1998). These claims may be true, but they go beyond normal macroeconomic microfoundations. By contrast, in this paper we show that the argument in the previous paragraph means that the conventional approach can be formally justified, using only the microfoundations available within a standard, simple new Keynesian framework.

In Section 3 we generalise the model to allow for the accumulation of public debt.[4] Endogeneity of public debt obviously leads to a need for endogenous fiscal policy if fiscal dominance is to be avoided: with two objectives to be satisfied (inflation and fiscal solvency) it is necessary to have two instruments (the interest rate and either public expenditure or the tax rate). Following an inflation shock, the rise in the interest rate caused by monetary policy leads to an accumulation of debt, as does the loss in tax revenues caused by the fall in output which follows the rise in the interest rate. Clearly fiscal policy needs to be tightened in these circumstances to ensure fiscal solvency. The important question concerns how active fiscal policy needs to be during the adjustment

[4]Endogenous accumulation of public debt is a normal feature in DSGE models (such as Woodford, 1996). This is because, as a matter of fact, tax rates cannot be immediately adjusted to offset whatever changes in revenue, or expenditure, actually happen.

process: should this policy merely ensure fiscal solvency or should it be actively involved in managing the economy?

Active fiscal policy will, in some circumstances be desirable. This is because in the first period, the expectations of the private sector are given and the policymaker may be able to exploit this fact. When the initial level of public debt is high, it will be optimal for the policymaker to choose to contract fiscally. This is because, if the interest rate was used to control inflation in the way suggested by the conventional wisdom, the result would be a large accumulation of public debt. As a consequence, the level of government spending would need to be greatly reduced in the new steady state to ensure public sector solvency, and that would be costly. In such circumstances, it can actually be optimal to *cut* the nominal interest rate in the first period so as to help reduce public debt. In these circumstances, government spending needs to fall sharply in the first period, *i.e.* fiscal policy needs to become actively involved in the stabilisation of the economy. By contrast, when the initial level of public debt is small, it is not helpful to act in this way, and one can continue to rely on the fact that monetary policy has a comparative advantage in controlling inflation. That is simply a consequence of the fact that raising the nominal interest rate to control inflation has little effect on the evolution of debt in these circumstances.

We show that these results are robust to including endogenous capital accumulation, and inflation persistence. These features are normally added to new Keynesian models in an attempt to make them more realistic (Examples include Dupor, 2001; Carlstrom and Fuerst, 2005; Christiano, Eichenbaum and Evans, 2005; Smets and Wouters, 2007 and Christiano, Trabandt and Walentin, 2011.) We show that introducing these additional features does not overturn our finding that fiscal policy should be active when public debt is initially high. Indeed these features provide additional reasons as to why it may be desirable for fiscal policy to do more than just passively ensure fiscal solvency. We also carry out some further robustness checks by varying the persistence of the inflation shock and varying the fraction of backward-looking price-setters in the economy.

With capital in the model, monetary policy affects investment decisions as well as consumption decisions; having an active fiscal instrument will thus influence the path taken by capital – since it will influence the path taken by the interest rate. Such a change in the time path of the capital stock will have an influence on the process of disinflation through its effects on the supply side of the economy. We will show that having an active fiscal instrument can moderate the reduction in the capital stock which occurs along the adjustment path because of the higher interest rate, and so can reduce the welfare loss which any such reduction in capital causes to happen. But we show that, given a realistic parameterisation of the model, the effects of this are small.

Allowing price setters to be, at least in part, backward-looking has also become an important feature of the DSGE modeling tradition. In the simplest New Keynesian Phillips curve, inflation is entirely forward-looking. But empirical evidence shows that such a setup does not adequately match the observed inflation data, which shows a high degree of persistence (Mankiw, 2001; Roberts, 2001). Following Fuhrer and Moore (1995), it has become conventional to make use of a mixed backward/forward-looking Phillips curve.[5] Introducing such inflation persistence into the Phillips curve can make it desirable for fiscal policy to share the burden of inflation control with monetary policy. The fundamental idea here is that, since we assume that backward-looking price setters do not optimise, there is no labour-supply response to higher interested rates. This means that, for these individuals, the comparative-advantage reason for concentrating on monetary policy is no longer valid. This means that there is a reason for using fiscal restraint as part of a disinflationary process.

1.2 Related Literature

A number of recent papers consider optimal monetary and fiscal policy, in models with sticky prices.

Schmitt-Grohe and Uribe (2007) consider a medium-scale model with capital accumulation and a number of real and nominal rigidities. They use the numerical second-order perturbation technique developed in Schmitt-Grohe and Uribe (2004a) to find the Ramsey-optimal policy. In this policy the tax-rate feedback rules have only a small response to any fluctuation in government debt; and fiscal policy does not appear to be

[5] There has been a large subsequent literature exploring the robustness of the Fuhrer and Moore finding, using two contrasting approaches. Some researchers construct simple semi-structural systems and estimate inflation together with other macroeconomic variables in a vector-auto-regressive system. Using this approach, Fuhrer (2010) concludes that inserting inflation persistence into the Phillips curve makes a significant contribution to the explanation of inflation. Other researchers use Bayesian methods to estimate models that incorporate multiple forms of persistence. Studies using this method usually find lower degree of backward-lookingness. (See Smets and Wouters, 2007; Amisano and Tristani, 2010 for instance.) As a result of this disagreement, the empirical importance of lagged inflation in the Phillips curve remains an unresolved issue. Because of this it is important to investigate the coordination of fiscal policy and monetary policy in a model in which inflation persistence is present.

actively engaged in stabilising the economy. But the authors do not derive their optimal policy analytically and hence the reason for their result is not very clear. Benigno and Woodford (2004) develop a linear-quadratic approach to study optimal policy. They also find that it is desirable for government debt to follow a unit root process, with no active feedback from the level of debt to the setting of fiscal policy. Both of these papers treat government spending as exogenous and use income taxation as the fiscal policy instrument. However, Ricardian equivalence is a property of the models of consumer behaviour used in both papers, something which obviously biases such investigations against the active use of fiscal policy. Here we instead treat the tax rate as exogenous and focus on the possibility of adjustments to government spending, the reason being that – as is well known[6] – the effects on aggregate demand of temporary changes in government spending are not neutralised by the effects of Ricardian equivalence.

Eser (2006), Stehn and Vines (2007, 2009), Cheng (2008), Eser, Leith and Wren-Lewis (2009) (henceforth, ELWL) and Leith and Wren-Lewis (2013) all study the coordination of government spending and monetary policy in models with sticky prices. All of these papers underpin the present work. Eser (2006) contains the first demonstration of the conventional view; the relevant result was first presented publicly in a discussion paper by ELWL, and published formally by Leith and Wren-Lewis (2013). We restate their proof and provide a diagrammatic intuition. In the presence of debt, ELWL and Leith and Wren-Lewis (2013) show that in all periods other than the first, government spending will deviate from the optimal steady state only to the extent required to ensure fiscal solvency, and that additional fiscal stabilisation may be valuable in the first period. But they do not show that, if the initial level of debt is high, a very large fiscal contraction may be required, coupled with a reduction in the nominal interest rate.[7] Stehn and Vines (2007) had already shown two years previously that if the initial level of public debt is high, then the optimal initial movements of both policy instruments will be very far away from the conventional wisdom, but they provided no good analytical reason for this result. Cheng (2008) finds – as we do – that the effect of including endogenous capital accumulation has very little effect in shifting the optimal policy from the conventional wisdom, but does not provide the relevant analytical reason. ELWL show that the conventional wisdom remains true even when allowing for inflation inertia in price and wage-setting, but, they do not consider the case in which some price-setters respond to the lagged output gap,

[6] See Barro (1979) for example.

[7] In both of these papers, the authors fail to notice the central idea of this paper. ELWL did not conduct any numerical simulations. Leith and Wren-Lewis (2013) did not focus on an inflation shock.

which we show below to be crucial.[8]

1.3 Plan of this Paper

The paper proceeds as follows. In Section 2 we use the reduced form of a simple, standard New Keynesian model to show why government spending plays no role in the optimal policy under commitment in this model. In Section 3, we formally set up a microfounded New Keynesian model with endogenous public debt. We derive the first-order approximated system in gap form and the second-order approximated welfare function. We study the optimal policy for this system analytically and explore the implications of these analytical findings by simulating our system numerically. Section 4 shows that our results are robust to the inclusion of endogenous capital accumulation and inflation persistence. Section 5 provides further robustness results to other parameterisations. Section 6 concludes.

2 Optimal Policy in a Simple New Keynesian Model

In this section, we use the reduced-form model of the simplest possible New Keynesian model which includes public expenditure, financed by lump-sum taxes, to show that fiscal policy plays no role in stabilisation when the policymaker can commit to a time-inconsistent policy.[9]

The reduced-form system is:

$$\begin{aligned}
\sigma c_t &= \sigma E_t c_{t+1} - (r_t - E_t \pi_{t+1}), && (1)\\
\pi_t &= \beta E_t \pi_{t+1} + \kappa(mc_t + \hat{\mu}_t), && (2)\\
y_t &= \theta c_t + (1-\theta) g_t, && (3)\\
mc_t &= \sigma c_t + \varphi y_t. && (4)
\end{aligned}$$

where, c_t denotes consumption, g_t denotes government spending, y_t denotes output, r_t is the nominal interest rate and π_t is inflation. All these variables are log-deviations from the flex-price economy. $\hat{\mu}_t$ is a cost-push shock. All parameters are positive. These four equa-

[8]Stehn and Vines (2007) and Leith and Wren-Lewis (2013) also consider policymaking when the policymaker acts with discretion, but we do not, in this paper, consider such policies.

[9]The model is essentially the one first introduced by Clarida, Gali and Gertler (1999), with fiscal policy added. In the next section we introduce a DSGE model characterised by agents making optimal decisions and show that the model and the loss function written down here are special cases of that system.

tions are the consumption Euler equation, New-Keynesian Phillips curve, goods market clearing condition and an equation for the marginal cost respectively. A key assumption in this system is that the government has access to a lump-sum transfer so that its budget is always balanced.

The microfounded social loss function is

$$L_t = \frac{1}{2} E_t \sum_{s=t}^{\infty} \beta^{s-t} l_s, \tag{5}$$

where

$$l_s = \frac{\epsilon}{\kappa}\pi_s^2 + \varphi y_s^2 + \sigma\theta c_s^2 + \sigma(1-\theta) g_s^2. \tag{6}$$

The consumption Euler equation is not binding because the nominal interest rate can be used freely to control consumption. We use the goods market clearing condition to rewrite the Phillips curve as follows:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa((\sigma + \varphi\theta)c_s + \varphi(1-\theta)g_s) + \kappa\hat{\mu}_s. \tag{7}$$

The Lagrangian for the policymaker is:

$$\begin{aligned} \mathscr{L}_t &= E_t \sum_{s=t}^{\infty} \beta^{s-t} [\frac{1}{2}\left(\frac{\epsilon}{\kappa}\pi_s^2 + \varphi(\theta c_s + (1-\theta)g_s)^2 + \sigma\theta c_s^2 + \sigma(1-\theta)g_s^2\right) \\ &\quad + \lambda_{s+1}^{\pi}(\pi_s - \beta\pi_{s+1} - \kappa((\sigma+\varphi\theta)c_s + \varphi(1-\theta)g_s) - \kappa\hat{\mu}_s)]. \end{aligned} \tag{8}$$

Under commitment, the first order conditions are

$$\frac{\partial \mathscr{L}_t}{\partial \pi_t} = \frac{\epsilon}{\kappa}\pi_t + \lambda_{t+1}^{\pi} - \lambda_t^{\pi} = 0, \tag{9}$$

$$\frac{\partial \mathscr{L}_t}{\partial c_t} = \sigma\theta c_t + \theta\varphi y_t - \kappa(\sigma+\varphi\theta)\lambda_{t+1}^{\pi} = 0, \tag{10}$$

$$\frac{\partial \mathscr{L}_t}{\partial g_t} = \sigma(1-\theta)g_t + (1-\theta)\varphi y_t - \kappa\varphi(1-\theta)\lambda_{t+1}^{\pi} = 0. \tag{11}$$

The optimal fiscal and monetary policies are

$$c_t = \frac{\kappa}{\theta}\lambda_{t+1}^{\pi} - \frac{\varphi(1-\theta)}{(\sigma+\varphi\theta)} g_t, \tag{12}$$

$$g_t = \frac{\kappa\varphi}{\sigma+\varphi(1-\theta)}\lambda_{t+1}^{\pi} - \frac{\varphi\theta}{\sigma+\varphi(1-\theta)} c_t. \tag{13}$$

We solve for c_t and g_t to get:

$$c_t = \frac{\kappa}{\theta}\lambda^{\pi}_{t+1}, \quad g_t = 0. \tag{14}$$

where λ^{π}_{t+1} is the Lagrange multiplier for the Phillips curve which measures the marginal social loss of raising inflation by one unit. The key result is that government spending is not involved in the optimal stabilisation policy under commitment.[10] Setting $\lambda^{\pi}_0 = 0$, the full optimal commitment policy is given by:

$$c_t = \begin{cases} -\frac{\epsilon}{\theta}\pi_t, & \text{for } t = 0. \\ c_{t-1} - \frac{\epsilon}{\theta}\pi_t, & \text{for } t \geq 1. \end{cases}, \qquad g_t = 0.$$

To understand why fiscal policy does not move, first, note that although the optimal commitment policy is dynamic, the choice between monetary policy and fiscal policy in each period is static because both the period loss function and the marginal cost are static. We may separate the policymaker's task into two parts. One of these is to choose a dynamic path of inflation that minimises the loss intertemporally. And the other part is to find the least costly combination of c_t and g_t which achieves a given level of inflation.

The choice of c_t and g_t can be understood diagrammatically as in Figure 1. Each Phillips curve (represented by a red dashed line) indicates a combination of consumption and government spending that leads to a particular level of inflation at time t. The slope of the Phillips curve in (c, g)-space is a constant:

$$\frac{dc_t}{dg_t}|_{\pi_t = const} = -\frac{\varphi(1-\theta)}{\sigma + \varphi\theta}. \tag{15}$$

The iso-loss ellipse (represented by a blue line in Figure 1) for a given level of inflation shows the combination of consumption and government spending that leads to a particular level of social loss. The slope of the iso-loss ellipse is:

$$\begin{aligned} \frac{dc_t}{dg_t}|_{l_t = const} &= -\frac{(1-\theta)\sigma g_t + \varphi(1-\theta)y_t}{\sigma\theta c_t + \theta\varphi y_t}, \\ &= -\frac{\varphi(1-\theta)}{\sigma + \varphi\theta} \times \left(\frac{c_t + \Xi_1 g_t}{c_t + \Xi_2 g_t}\right). \end{aligned} \tag{16}$$

[10] In fact, in this simple model, when the policymaker acts under discretion, government spending is not involved stabilisation as well. By setting $E_t\pi_{t+1} = 0$, the first order condition for inflation becomes $\lambda^{\pi}_{t+1} = -(\epsilon/\kappa)\pi_t$ and the optimal discretion policy is $c_t = -(\epsilon/\theta)\pi_t, g_t = 0$.

where

$$\Xi_1 = \frac{\sigma + \varphi(1-\theta)}{\theta\varphi}, \quad \Xi_2 = \frac{(1-\theta)\varphi}{\sigma + \varphi\theta}, \quad \Xi_1 - \Xi_2 = \frac{\sigma(\sigma+\varphi)}{\theta\varphi(\sigma+\varphi\theta)} > 0.$$

We seek an outcome in which the slope of the iso-loss ellipse is tangent to the Phillips curve. Since $\Xi_1 > \Xi_2$ this can only happen when $g_t = 0$, as shown in Figure 1. We now use a *reductio-ad-absurdum* argument to explain the reason for this.

Suppose that a disinflation policy had achieved the optimal level of inflation at any time t but had done this by reducing both consumption and government spending. We know that, at such a point, the slope of the iso-loss ellipse is more negative than the slope of the Phillips curve. This means that by marginally raising government spending and reducing consumption, inflation could be kept unchanged, but the social loss could be reduced. That is true because, as described at the beginning of this paper, consumption has a comparative advantage over government spending in the control of inflation. Such a marginal increase in government expenditure will increase welfare whenever government spending is negative; this is why it is optimal to set $g_t = 0$.[11]

The first row of Table 1 provides a summary of the analytical results of this section; we will refer to this table repeatedly in what follows.

[Insert Table 1 here.]

3 Optimal Fiscal and Monetary Policy in a New-Keynesian Model with Endogenous Public Debt

In this section we set out a standard New-Keynesian model with optimising agents, and with endogenous public debt, presented by ELWL and Leith and Wren-Lewis (2013). We use this model to show that the optimal response to an inflation shock can involve fiscal contraction.[12] We will show that the simple reduced form model used in the previous section can be derived from the model presented in this section. In Appendix A.1 we present a more complex version of our model, in which the model is generalised to include

[11] A similar argument is made in ELWL.

[12] The government uses a lump-sum tax to finance a steady-state subsidy, which ensures that the steady-state is efficient. With this assumption, we set aside the problem of inflation bias caused by inefficiently low output in the steady state as a result of monopolistic competition and distortionary taxes.

both capital accumulation and inflation persistence. That more general version is used in the robustness exercises which follow this section.

3.1 The Model

3.1.1 Consumers

Consumers are homogeneous. We assume that consumers have an additively separable utility which is increasing in consumption C_t and government spending G_t and decreasing in labour N_t:

$$U_t = E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\frac{C_s^{1-\sigma}}{1-\sigma} + \chi \frac{G_s^{1-\sigma}}{1-\sigma} - \frac{N_s^{1+\varphi}}{1+\varphi} \right), \tag{17}$$

where E_t is the expectation operator conditional on the information at time t. Consumers supply labour, consume, save by buying government bonds and make capital investments. The budget constraint of the representative consumer is as follows:

$$P_t C_t + E_t(Q_{t,t+1} D_{t+1}) = D_t + (1-\tau)(W_t N_t + \Omega_t) - T_t,$$

where P_t and W_t denote the price level and the nominal wage respectively. We assume τ is an exogenous income tax rate. Ω_t is the profit from the firms and T_t is a lump-sum tax to finance a production subsidy which offsets distortions caused by monopolistic competition. This is a common short-cut to ensure that the steady state of the system is efficient. $Q_{t,t+1}$ is the stochastic discount factor which determines the price to the consumer to transfer a state-contingent amount D_{t+1} of wealth from period t to $t+1$. Its relation to the nominal interest rate is $E_t(Q_{t,t+1}) = R_t^{-1}$. The usual transversality condition applies.

Consumers maximise utility subject to the budget constraint. The first order conditions are as follows:

$$1 = \beta R_t E_t \left(\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right), \tag{18}$$

$$C_t^{\sigma} N_t^{\varphi} = (1-\tau) \frac{W_t}{P_t}. \tag{19}$$

The first equation is the consumption Euler equation. The second equation is the intratemporal optimal condition which equates the marginal rate of substitution between consumption and leisure with the after-tax real wage.

3.1.2 Final Goods Firm

On the production side, there is a final goods firm which mixes intermediate goods using a Dixit-Stiglitz (1977) aggregator:

$$Y_t = \left(\int_0^1 Y_t(i)^{\frac{\epsilon_t - 1}{\epsilon_t}} di \right)^{\frac{\epsilon_t}{\epsilon_t - 1}},$$

and the demand for each type of good is:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon_t} Y_t,$$

where $P_t \equiv \left[\int_0^1 P_t(i)^{1-\epsilon_t} di \right]^{\frac{1}{1-\epsilon_t}}$.

3.1.3 Intermediate Goods Firm

Intermediate goods firms are assumed to be monopolistically competitive. Each firm i produces with labour. The production function for firm i is

$$Y_t(i) = A_t N_t(i).$$

where the productivity A_t is an exogenous AR(1) process such that $\hat{A}_{t+1} = \rho_A \hat{A}_t + \xi_{A,t+1}$. Aggregate labour is defined as $N_t \equiv \int_0^1 N_t(i) di$.

The aggregate production function is

$$Y_t = A_t N_t \Delta_t^{-1}, \tag{20}$$

where $\Delta_t \equiv \int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon_t} di$ is the price dispersion. When prices are more dispersed, that is when $Var_i(P_t(i))$ rises, the price dispersion Δ_t also increases. Conditional on the aggregate labour, the aggregate output decreases. Price distortion causes some firms to produce more than the others and hence reduces welfare relative to the flexible price equilibrium.

3.1.4 Price Setting

We model price setting as in Calvo (1983). Firms recalculate their prices with fixed probability $(1 - \gamma)$. With probability γ prices are not recalculated and are assumed to rise at the average rate of inflation. The forward-looking price-setters solve the first order conditions for profit maximisation and obtain the optimal solution. The price setting problem is:

$$\max_{P_t(i)} E_t \sum_{s=0}^{\infty} \gamma^s Q_{t,t+s} \left(P_t(i) Y_{t+s}(i) - \frac{1}{\mu_w} MC_{t+s}^n Y_{t+s}(i) \right),$$

where $MC_t^n = W_t/A_t$ is the nominal marginal cost and μ_w is a subsidy from the government which ensures that the steady-state is efficient. This required subsidy is $\mu_w = \epsilon/(\epsilon-1)(1-\tau)$, and is paid for by a lump sum tax T_t. Since there are no state variables in the price-setting problem, all forward-looking price-setters choose the same price P_t^{FL} to satisfy the following first order condition:

$$\frac{P_t^{FL}}{P_t} = \frac{E_t \sum_{s=0}^{\infty} \gamma^s Q_{t,t+s} \epsilon_{t+s} \left(\frac{P_{t+s}}{P_t}\right)^{\epsilon_{t+s}+1} \left(\frac{W_{t+s}}{P_{t+s} A_{t+s}}\right) Y_{t+s}}{E_t \sum_{s=0}^{\infty} \gamma^s Q_{t,t+s} (\epsilon_{t+s}-1) \left(\frac{P_{t+s}}{P_t}\right)^{\epsilon_{t+s}} Y_{t+s}}. \tag{21}$$

The evolution of the aggregate price is as follows:

$$P_t = \left(\gamma P_{t-1}^{1-\epsilon_t} + (1-\gamma) \left(P_t^{FL}\right)^{1-\epsilon_t} \right)^{\frac{1}{1-\epsilon_t}}. \tag{22}$$

The price dispersion, Δ_t, evolves as follows:

$$\Delta_t = \gamma \Pi_t^{\epsilon_t} \Delta_{t-1} + (1-\gamma) + \left(\frac{P_t^{FL}}{P_t} \right)^{-\epsilon_t}. \tag{23}$$

where $\Pi_t = P_t/P_{t-1}$. We define $\mu_t \equiv \epsilon_t/(\epsilon_t - 1)$ and assume an AR(1) process for the cost-push shock, *i.e.* $\hat{\mu}_{t+1} = \rho_\mu \hat{\mu}_t + \xi_{\mu,t+1}$.

3.1.5 The Government and Goods Market Clearing

The government buys goods, taxes income with a constant tax rate τ and issues a nominal debt $\bar{B}$. In each period, the production subsidy is fully funded by the lump-sum tax. The evolution of the nominal debt is given by:

$$B_{t+1} = R_t \left(B_t \frac{P_{t-1}}{P_t} + G_t - \tau Y_t \right). \tag{24}$$

where $B_{t+1} = \bar{B}_{t+1}/P_t$ is the real debt. The goods market clearing condition is given by:

$$Y_t = C_t + G_t. \tag{25}$$

In the steady state assume $C = \theta Y$ and $G = (1-\theta)Y$, where θ is the share of private consumption in output.

This completes the description of the model. In sum, the model consists of 8 equations (18), (19), (20), (21), (22), (23), (24) and (25). Together with the policy rules of the nominal interest rate and government spending, this system of equations enables us to solve for the following endogenous variables: $\{C_t, N_t, \frac{W_t}{P_t}, Y_t, \Pi_t, \frac{P_t^{FL}}{P_t}, B_t, \Delta_t, R_t, G_t\}$.

3.2 The Case of Flexible Price and the System in 'Gap' Form

In order to study the welfare loss caused by the distortions associated with nominal rigidities, we wish to write the system with nominal rigidities in the form of deviation from the efficient flexible price level. To do this, we first study the case with flexible price.

In the case of flexible price, the social planner maximises utility, Equation (17), subject to the goods market clearing condition and the production function. The first order conditions are:

$$(C_t^n)^{-\sigma} = \chi\,(G_t^n)^{-\sigma}, \tag{26}$$

$$(C_t^n)^{\sigma}\,(N_t^n)^{\varphi} = (1-\alpha)\frac{Y_t^n}{N_t^n}, \tag{27}$$

where the superscript 'n' denotes the efficient flexible price level. Equation (26) equates the marginal utility of consumption and government spending. Equation (27) equates the marginal rate of substitution between consumption and labour and the marginal product of labour.

These first order conditions, together with the goods market clearing condition and the production function with price dispersions $\Delta = 1$ form a system which, given the exogenous productivity shock process, describes the behaviour of $\{C_t^n, N_t^n, Y_t^n, G_t^n\}$. The natural interest rate R_t^n has to equal to the ratio of marginal utilities of consumption so that $R_t^n = \beta^{-1}(C_t^n)^{-\sigma}/E_t(C_{t+1}^n)^{-\sigma}$.

The model with nominal rigidities can be written in terms of the deviations from the natural levels as follows (Define $x_t \equiv \hat{X}_t - \hat{X}_t^n$, and $\hat{X}_t \equiv \log(X_t/X)$):

$$c_t = E_t c_{t+1} - \sigma^{-1}\left(r_t - E_t \pi_{t+1}\right), \tag{28}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa(\sigma c_t + \varphi y_t + \hat{\mu}_t), \tag{29}$$

$$y_t = \theta c_t + (1-\theta) g_t, \tag{30}$$

$$\hat{B}_{t+1} = r_t + \frac{1}{\beta}\left(\hat{B}_t - \pi_t + \frac{(1-\theta)}{B} g_t - \frac{\tau}{B} y_t\right) + \hat{\nu}_t. \tag{31}$$

where $\pi_t = \log \Pi_t$ is the inflation rate and $\hat{\nu}_t = -(\sigma(1-\rho_A) + \frac{1-\beta}{\beta})\frac{1+\varphi}{\sigma+\varphi}\hat{A}_t$. The parameter B denotes the steady-state debt to output ratio.

Notice that when government spending is financed by lump-sum tax, the evolution for debt, Equation (31), disappears. The system collapses back to Equations (1) – (4) in Section 2.

3.3 The Social Welfare Function

We derive a welfare-based loss function for the policymaker using a second-order approximation of the utility function around the efficient steady state. The derivation is given in Appendix A.3. The loss function is

$$L_t = -\frac{C^{1-\sigma}}{2\theta} E_t \sum_{s=t}^{\infty} \beta^{s-t}\left(\sigma\theta c_s^2 + \sigma(1-\theta) g_s^2 + \varphi y_s^2 + \frac{\epsilon}{\kappa}\pi_s^2\right) + t.i.p + O(3). \tag{32}$$

In the subsections which follow, we will make use of this loss function and the constraints of the economy (Equations (28) - (31)) to derive the optimal monetary and fiscal policy when the policymaker acts under commitment. First we explain how the model is calibrated.

3.4 Calibration

We let each period in the model be a quarter. The impatience parameter, β, is set to be 0.99 so that the steady state nominal interest rate is 4.2%. We set the share of consumption and government spending to output ratio in steady state to 0.75 and 0.25 respectively in the model. These parameters determine the relative preference for government spending in utility. The coefficient of relative risk aversion σ is 1, which implies logarithmic utility. The inverse of Frisch labour supply elasticity, φ, is 0.5. The steady state elasticity of substitution across varieties of consumption goods, ϵ, is set to 5. The probability of resetting price, γ is calibrated to be 0.75, which implies that prices are on average set once a year. These parameters are calibrated following the recent literature (*e.g.* Rotemberg and Woodford, 1997). We set the persistence of the inflation shocks to ρ_μ, and will explore other values our robustness checks.

Regarding public debt, we study economies with low, medium and high debt, which correspond to debt-to-output ratios of $0.025, 0.6$ and 1.5 respectively. (Since we have a quarterly model, these mean we set $B = 0.1, 2.4, 6$ respectively.) A 60% debt-to-output ratio is common across countries and is assumed Benigno and Woodford (2004) and Kirsanova and Wren-Lewis (2011). Some countries have debt-to-output ratios around 150%.[13] The tax rate τ is pinned down by the steady state debt to output ratio, *i.e.* $\tau = B(1-\beta) + (1-\theta)$. Parameter values are summarised in Table 2.

[Insert Table 2 here.]

3.5 Optimal Policy

We solve for the optimal coordination of fiscal and monetary policy when the policymaker rule is able to act with commitment.[14] The policymaker chooses r_t and g_t in order to minimise the loss function given by (32) subject to constraints (28)-(31).[15]

[13]Using World Bank data, the central government debt-to-output ratios in Japan, Greece and Italy in 2012 are $196\%, 163\%$ and 126% respectively.

[14]When the policymaker acts under discretion, after a shock the endogenous government debt will eventually need to return to its initial steady state, since any other outcome is time-inconsistent (Leith and Wren-Lewis, 2013). Consequently, government spending needs to adjust actively. Details of the results under discretion are available upon request.

[15]Details of the solution method is provided in Appendix A.4. We note that ELWL and Leith and Wren-Lewis (2013) have considered this model before. However, they do not provide the detailed results which follow. In particular, they do not provide a comparison between the outcomes in the low debt case, and the outcomes in the high debt case.

Specifically, the Lagrangian for the policymaker is:

$$
\begin{aligned}
\mathscr{L}_t &= E_t \sum_{t=s}^{\infty} \beta^{s-t} [\frac{1}{2} \left(\sigma\theta c_s^2 + \sigma(1-\theta) g_s^2 + \varphi y_s^2 + \frac{\epsilon}{\kappa}\pi_s^2 \right) \\
&+ \lambda_{s+1}^{\pi} \left(\pi_s - \beta\pi_{s+1} - \kappa \left((\sigma + \varphi\theta) c_s + (1-\theta)\varphi g_s \right) - \kappa \hat{\mu}_s \right) \\
&+ \lambda_s^{y} \left(\theta c_s + (1-\theta) g_s - y_s \right) \\
&+ \lambda_{s+1}^{c} \left(\sigma c_s - \sigma c_{s+1} + r_s - \pi_{s+1} \right) \\
&+ \lambda_{s+1}^{b} \left(r_s + \frac{1}{\beta} \left(\hat{B}_s - \pi_s + \frac{(1-\theta)}{B} g_s - \frac{\tau}{B} y_s \right) + \hat{\nu}_s - \hat{B}_{s+1} \right)].
\end{aligned}
$$

The first order conditions are:

$$
\begin{aligned}
\frac{\partial \mathscr{L}_t}{\partial \pi_t} &= \frac{\epsilon}{\kappa}\pi_t + \lambda_{t+1}^{\pi} - \lambda_t^{\pi} - \beta^{-1}\lambda_t^{c} - \beta^{-1}\lambda_{t+1}^{b} = 0, && (33) \\
\frac{\partial \mathscr{L}_t}{\partial y_t} &= \varphi y_t - \lambda_t^{y} - \frac{1}{\beta}\frac{\tau}{B}\lambda_{t+1}^{b} = 0, && (34) \\
\frac{\partial \mathscr{L}_t}{\partial c_t} &= \sigma\theta c_t - \kappa(\sigma + \varphi\theta)\lambda_{t+1}^{\pi} + \theta\lambda_t^{y} + \sigma\lambda_{t+1}^{c} - \beta^{-1}\sigma\lambda_t^{c} = 0, && (35) \\
\frac{\partial \mathscr{L}_t}{\partial \hat{B}_t} &= -\beta^{-1}\lambda_{t+1}^{b} + \beta^{-1}\lambda_t^{b} = 0, && (36) \\
\frac{\partial \mathscr{L}_t}{\partial r_t} &= \lambda_{t+1}^{c} + \lambda_{t+1}^{b} = 0, && (37) \\
\frac{\partial \mathscr{L}_t}{\partial g_t} &= \sigma(1-\theta) g_t - \kappa\varphi(1-\theta)\lambda_{t+1}^{\pi} + (1-\theta)\lambda_t^{y} + \frac{1}{\beta}\frac{(1-\theta)}{B}\lambda_{t+1}^{b} = 0. && (38)
\end{aligned}
$$

The optimal commitment policy is fully described by Equations (28) – (31) and (33) – (38), the shock process and the initial values for the state variables. The state variables of this system include $\hat{B}_t$ and the Lagrange multipliers for the jump variables $\lambda_t^{\pi}, \lambda_t^{c}$. Assume that the economy is in the interior steady state before any shock occurs which coincides with the flexible price equilibrium. Since the equations describing the private sector of the economy are not binding initially, the Lagrange multipliers λ_0^{π} and λ_0^{c} are set to zero to make sure that the initial policy is optimal.

Two things can be seen directly from these conditions. First, a rise in the nominal interest rate has the benefit of reducing current consumption, but has the cost of raising debt. The first order condition for the nominal interest rate states that these marginal social benefits and losses must be identical in the optimal policy. Second, the Lagrange multiplier for debt, λ_t^{b}, follows a random walk, consistent with Benigno and Woodford (2004) and Schmitt-Grohe and Uribe (2004b). The rationale for this is the following.

Suppose a shock drives government debt away from its pre-shock steady state. Since the period loss function is quadratic in government spending and there is discounting of future losses, the policymaker spreads the burden of debt reduction for an infinite number of periods, which means that the government spending gap stays below zero (and consumption above zero) infinitely, and the Lagrange multiplier for government debt does not go back to zero.[16] Taken together, these mean that $\lambda_t^b = \lambda_{t+1}^b = -\lambda_{t+1}^c$, for $t \geq 0$.

We solve for c_t and g_t, and using the fact that $\lambda_t^b = \lambda_{t+1}^b = -\lambda_{t+1}^c$, for $t \geq 0$ and $\lambda_0^c = 0$, and obtain:

$$\begin{aligned} g_t &= \frac{1}{\sigma+\varphi}\left(-\frac{\varphi}{\beta}+\varphi+\frac{\sigma(1-\tau)+\varphi\theta}{\sigma\beta B}\right)\lambda_0^b, \qquad \text{for } t \geq 1, \\ g_0 &= \frac{1}{\sigma+\varphi}\left(\varphi+\frac{\sigma(1-\tau)+\varphi\theta}{\sigma\beta B}\right)\lambda_0^b. \end{aligned} \tag{39}$$

Government spending for $t \geq 1$ follows a random walk. For plausible parameterisations, λ_0^b becomes negative when there is a positive inflation shock. Government spending falls in the first period, and subsequently moves to a negative level that ensures fiscal solvency and stays at this level permanently. This means that government spending is not directly involved in controlling inflation after the first period. Moreover, $g_t - g_0 = -\frac{\varphi}{\beta(\sigma+\varphi)}\lambda_0^b > 0$ when $\lambda_0^b < 0$, which means that the initial fall in government spending is larger than what is required to maintain fiscal solvency. This means that the first period is one in which government spending helps to limit the increase in government debt (Leith and Wren-Lewis, 2013).[17]

We now explore these findings empirically by considering the impulse responses to a 1% rise in inflation. These are shown in Figure 2. We compare this system with the system in Section 2 in which a lump-sum tax is available, which is indicated by the blue solid lines. The red, black and green line shows the system with low, medium and high debt (debt-to-output ratios of $0.025, 0.6$ and 1.5 in the initial steady state).

[16]The temptation for the policymaker to renege on the time-inconsistent policy to keep debt permanently above zero is measured by the Lagrange multiplier λ_t^b. If allowed to re-optimise along the adjustment path, the policymaker would reset $\lambda_t^b = 0$. This involves cutting the nominal interest rate unexpectedly to reduce debt and is inflationary. If expected, the price-setters will have raised their inflation expectation before the monetary easing, which makes inflation stabilisation harder. Leith and Wren-Lewis (2013) refer to this time-inconsistency problem with the debt stabilisation bias.

[17]Under Woodford (1999)'s timeless perspective policymaking, the initial fall in government spending will not be different from its long-run level.

When lump-sum taxes are used (blue line), the results are as presented in the previous section: government spending stays constant at zero. Inflation is controlled by raising the nominal interest rate alone. An increase in the nominal interest rate reduces output and consumption which helps to reduce current inflation.

When government debt is endogenous the results are different from this.

Consider first the response when there is a small level of government debt. The differences from the findings of the previous section turn out to be small. Apart from government spending, the results are strikingly similar to the previous case in which lump-sum taxes are adjusted to prevent any endogeneity of public debt. In response to a positive inflation shock, the nominal interest rate rises immediately to stabilise inflation and government spending falls. The short-run adjustment of the economy involves a fall in real activity and a rise in government debt. In the long run, the stock of public debt is permanently higher than its pre-shock level and government spending remains permanently lower, which stops government debt from increasing infinitely. Due to this wealth effect coming from the increase in the stock of government bonds, in the new steady state, consumption is higher, employment is lower and output is decreased. But these effects are tiny numerically, and that is why the results for consumption and output are very similar to the case with lump-sum taxes in Figure 2.

When the steady-state level of government debt is large and the inflation shock is persistent, the results for optimal policy are *very* different. Government spending falls sharply in the initial period and recovers in subsequent periods. The nominal interest rate actually falls initially in response to the positive inflation shock before rising in subsequent periods. The reason for this striking finding is as follows.

We know from Schmitt-Grohe and Uribe (2007) and Woodford (2003b) that debt can deviate permanently from the initial steady state when the policymaker acts under commitment. However, it does not mean that the policymaker does not care about the level of debt in the new steady state. Allowing a large permanent increase in public debt would mean that government spending had to be greatly curtailed in the new steady state, to ensure that fiscal solvency remained possible in the face of the large increase in debt interest payments. A policymaker has to find a balance between the short-run benefit of inflation stabilisation and the long-run cost of having a higher stock of debt and lower government spending. When the stock of public debt is already large in the initial steady-state, a rise in the nominal interest rate to control inflation will lead to a large rise in the level of public debt and the long-run costs with become important.

When government debt is already high, the optimal commitment policy involves inflating away part of the increase in debt – at least to some extent – even although this conflicts with the objective of controlling inflation. But more importantly, since changes in the nominal interest rate have a significant effect on the evolution of debt, it may be beneficial to reduce the nominal interest rate initially, so as to cut debt initially. Doing this sooner is better than doing this later, because an expectation of a future monetary expansion to inflate away debt in the future would have inflationary effects today, making current inflation harder to stabilise. Government expenditure needs to fall sharply in the first period, both to help stabilise public debt, and especially to prevent inflation from getting out of hand, even although - as we have seen - fiscal contraction is not as effective an instrument as monetary policy for the control of inflation.

The results of this section are summarised in the first two rows of Table 1.

4 Robustness I: Optimal Policy in the general New-Keynesian Model

In this section we make use of the more general New-Keynesian model with endogenous capital accumulation and inflation persistence, which is set out in Appendix A.1. We show that introducing these additional features do provide additional reasons why fiscal policy may act more active throughout the entire inflation stabilisation path. However, these effects are less important compared with the effects resulting from introducing a large stock of public debt.

In the following, we present the the log-linearised system and the microfounded second-order approximated loss function in 'gap' form. [18] The reduced form system is:

$$
\begin{aligned}
c_t &= E_t c_{t+1} - \sigma^{-1}\left(r_t - E_t\pi_{t+1}\right), && (40)\\
E_t(\Delta k_{t+1}) &= \beta E_t(\Delta k_{t+2}) + \frac{1}{J''}\left(r_t - E_t\pi_{t+1}\right)\\
&\quad - \frac{(1-(1-\delta)\beta)}{J''} E_t\left(\frac{1+\varphi}{1-\alpha} y_{t+1} + \sigma c_{t+1} - \frac{1+\varphi\alpha}{1-\alpha} k_{t+1}\right) && (41)\\
\pi_t &= \chi^{FL}\beta E_t\pi_{t+1} + \chi^{BL}\pi_{t-1} + \kappa_c c_t + \kappa_{y0} y_t + \kappa_{y1} y_{t-1} - \kappa_k k_t + \kappa_c \hat{\mu}_t, && (42)\\
y_t &= \theta_c c_t + \theta_i\left(\frac{1}{\delta} k_{t+1} - \frac{(1-\delta)}{\delta} k_t\right) + \theta_g g_t, && (43)\\
\hat{B}_{t+1} &= r_t + \frac{1}{\beta}\left(\hat{B}_t - \pi_t + \frac{\theta_g}{B} g_t - \frac{\tau}{B} y_t\right) + \hat{\nu}_t. && (44)
\end{aligned}
$$

where k_t denotes the capital gap. α denotes the share of capital in the production function, δ is the capital depreciation rate and $J'' < 0$ is the curvature of a Hayashi (1982) type capital adjustment costs, $\theta_c, \theta_i, \theta_g$ denote the share of consumption, investment and government spending to output in the steady state. We follow Neiss and Nelson (2003) and define the natural level of output such that the natural level of output in a given period depends on the capital stock consistent with the flexible-price model (rather than the capital stock that actually exists in the economy in that period, as proposed by Woodford (2003a)). Edge (2003) show that with this assumption, the second-order approximated utility-based welfare function can be completely characterised in terms of the 'gap' variables. The Phillips curve now follows Steinsson (2003) which assumes there are forward-looking and backward-looking price-setters in the economy. Specifically, with fixed probability $(1-\gamma)$ firms can reset price. If prices are recalculated then a proportion ω of the price resetting agents use a backward-looking rule of thumb to set their price and proportion $(1-\omega)$ calculate the optimum price. With probability γ prices are not recalculated and are assumed to rise at the average rate of inflation. Following Steinsson (2003), the backward-looking price-setters respond to the lagged output gap with sensitivity parameter η. The Phillips curve parameters $\chi^{FL}, \chi^{BL}, \kappa_c, \kappa_{y0}, \kappa_{y1}, \kappa_k, \kappa_\mu$ are all presented in Appendix A.2.

The microfounded loss function is:

$$
L_t = -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} l_s + t.i.p + O(3), \qquad (45)
$$

[18] Appendix A.1 provides detailed derivations of both the model and the loss function.

where

$$
\begin{aligned}
l_s &= \sigma\theta_c c_s^2 + \sigma\theta_g g_s^2 + \varphi(1-\alpha)n_s^2 + \alpha(1-\alpha)(n_s - k_s)^2 - \frac{\theta_i}{\delta}J''\left(\Delta k_{s+1}\right)^2 \\
&\quad + \underbrace{\frac{\epsilon}{\kappa}\pi_s^2 + 2\frac{\epsilon}{\kappa}\phi_1\left(\Delta\pi_s\right)y_{s-1} + \frac{\epsilon}{\kappa}\phi_2\left(\Delta\pi_s\right)^2 + \frac{\epsilon}{\kappa}\phi_3 y_{s-1}^2}_{Var_i(\hat{P}_t(i))}, \qquad (46)
\end{aligned}
$$

and where parameters ϕ_1, ϕ_2, ϕ_3 are defined in the Appendix A.3.

The loss function contains quadratic terms in consumption and government spending as before. The third term is a quadratic term in labour.[19] The fourth term is a quadratic term in the capital to labour ratio. This term reflects the fact that when the capital increases, the marginal product of labour rises, so it will be desirable to work more. The fifth term is related to capital adjustment costs. Inefficient adjustment of the capital stock wastes resources due to adjustment costs and this reduces welfare. The second line in the loss function shows losses coming from price dispersion. This contains the standard quadratic term in current inflation (Woodford 2003a). Steinsson (2003) shows that when backward-looking price-setters are present ($\omega > 0$), the price dispersion depends on lagged inflation. Furthermore, when backward-looking price-setters respond to lagged output ($\eta > 0$), the lagged output will appear in the price dispersion.

4.1 The Effects of Endogenous Capital Accumulation

This subsection studies the effect of adding the endogeneity of capital to the model in 3. We to assume, for now, that all price-setters are forward-looking ($\omega = 0$).

We first assume – unrealistically – that there are no capital adjustment costs $J'' = 0$, so that we can isolate the effect of capital accumulation in this analysis. In this case, the period loss function l_t is static, and the Phillips curve is given by

$$\pi_t = \beta E_t \pi_{t+1} + \kappa_c \left(c_t + \frac{1}{\sigma}\frac{\alpha+\varphi}{1-\alpha}y_t - \frac{\alpha(1+\varphi)}{\sigma(1-\alpha)}k_t + \frac{1}{\sigma}\hat{\mu}_t \right)$$

where the terms inside the bracket are static (since the stock of capital, k_t, is predetermined). So, as in Section 2, the tradeoff between c_t and g_t is still given by the tangency condition between the marginal cost curve and the 'iso-loss' ellipse in (c, g) space. The

[19]This term was not explicitly present in the previous loss function; the linear production function in the simple model enable the labour term to be replaced by the output gap.

slope of the Phillips curve in (c, g) space is given by:

$$\frac{dc_t}{dg_t}|_{\pi_t=const} = -\frac{\frac{\alpha+\varphi}{1-\alpha}\theta_g}{\sigma + \frac{\alpha+\varphi}{1-\alpha}\theta_c}. \tag{47}$$

The slope of the iso-loss ellipse is:

$$\frac{dc_t}{dg_t}|_{l_t=const} = -\frac{\sigma\theta_g g_t + \frac{1}{1-\alpha}\left((\alpha+\varphi)y_t - \alpha(1+\varphi)k_t\right)\theta_g}{\sigma\theta_c c_t + \frac{1}{1-\alpha}\left((\alpha+\varphi)y_t - \alpha(1+\varphi)k_t\right)\theta_c}. \tag{48}$$

When capital accumulation is endogenous, k_t appears in the social loss function and therefore in the slope of the iso-loss ellipse. The reason is as follows. In response to a positive inflation shock, if the policymaker disinflates only through monetary policy, then, other things being equal, consumption falls and labour supply goes up. However, with a predetermined stock of capital, the marginal product of labour must fall if the extra labour is to be employed, and using labour inefficiently in this way is suboptimal. Conditional on achieving a particular level of inflation, it would improve welfare somewhat if there were less of a reduction in the capital stock. That could be brought about by reducing the strength of monetary contraction but, at the same time, introducing some fiscal contraction. As k_t adjusts over time, the degree to which fiscal policy is employed in this way, to moderate the labour supply effects, will vary along the transition path. This overall conclusion is that, analytically, including endogenous capital accumulation brings about adjustments in government spending along the optimal path.

Figure 3 shows the impulse responses when we simulate the model with endogenous capital accumulation. We set $J'' = -10, \alpha = 0.34, \delta = 0.025, \theta_c = 0.56, \theta_g = 0.2$. Lines with different colours represent different debt-to-output ratios. Three things are apparent in Figure 3. First, the debt-to-output ratio remains the key determinant of the monetary and fiscal policies. Second, even when the debt-to-output ratio is high at 150% (green lines), the nominal interest rate still rises initially. The reason is that raising the nominal interest rate now reduces capital and output, and this of itself has a disinflationary effect. Third, from period 2 up to the long run, government spending moves only slightly and the movement is hardly observable in the figure.

It is possible to give an analytical justification for these empirical findings. We can rewrite the slope of the iso-loss ellipse using the log-linearised capital accumulation equation and the goods market clearing condition as follows:

$$\frac{dc_t}{dg_t}|_{l_t=const} = -\frac{\theta_g}{\theta_c} \times \frac{\sigma g_t + \frac{\alpha+\varphi}{1-\alpha}(\theta_c c_t + \theta_g g_t) + \frac{\alpha+\varphi}{1-\alpha}\left(\frac{\theta_i}{\delta}\Delta k_{t+1} + (\theta_i - \alpha(1+\varphi))k_t\right)}{\sigma c_t + \frac{\alpha+\varphi}{1-\alpha}(\theta_c c_t + \theta_g g_t) + \frac{\alpha+\varphi}{1-\alpha}\left(\frac{\theta_i}{\delta}\Delta k_{t+1} + (\theta_i - \alpha(1+\varphi))k_t\right)}.$$

We note two points. First, if capital is fixed, the iso-loss ellipse has the same slope as the Phillips curve when $g_t = 0$, so government spending is inactive. Second, since the capital stock evolves gradually, the term related to capital is likely to be small. That is why adding capital has only a small effect on g_t.

The results of this subsection are summarised in the third row of Table 1.

4.2 The Effects of Inflation Persistence

Next, we examine the tradeoff between consumption and government spending in the presence of all of endogenous public debt, endogenous capital and backward-looking price-setters. Because of inflation persistence, inflation is no longer simply driven by the marginal cost; lagged inflation and the output gap also affect inflation in the current period. The model is now rather complex.

Nevertheless, it is still possible to obtain some analytical insights by first considering a simplifying case in which the backward-looking price-setters do not respond to lagged output ($\eta = 0$). Under this circumstance, current inflation depends not only on expected future inflation, but also on past inflation:

$$\pi_t = \beta\chi^{FL} E_t \pi_{t+1} + \chi^{BL}\pi_{t-1} + \kappa_c \left(c_t + \frac{1}{\sigma}\frac{\alpha+\varphi}{1-\alpha} y_t - \frac{\alpha(1+\varphi)}{\sigma(1-\alpha)} k_t + \frac{1}{\sigma}\hat{\mu}_t\right).$$

While this addition alters the optimal disinflation path intertemporally, the intratemporal tradeoff between monetary and fiscal policy is not affected because the relative efficiency of these policies remain unchanged. In fact, the slopes of the Phillips curve and the iso-loss ellipse in (c, g) space are identical to the model with the New Keynesian Phillips curve, given by Equation (47) and Equation (48) respectively. As a result, we can conclude that inflation persistence does not, of itself, make government spending adjust along the transition path.

Figure 4 presents the impulse responses to the system when inflation persistence is added. We assume the share of backward-looking price-setters is $\omega = 0.75$, but also suppose that $\eta = 0$. Two things can be seen from this figure. First, when backward-looking price-setters are present, inflation is persistent and remains positive along the adjustment path. [20] Second, in line with the argument of the previous paragraph, introducing backward-looking price-setters itself does not make government spending more active after the first period.

However, when lagged output influences the way in which backward-looking price-setters adjust prices as in Steinsson (2003), government spending moves along the entire adjustment path. The reason is that fiscal and monetary policy affects the lagged output gap which in turn affects the price set by the backward-looking price-setters. Since the backward-looking price-setters do not optimise, the comparative advantage argument can no longer be applied to the backward-looking price-setters. This means that it is optimal to cut government expenditure as part of the process of bringing down inflation.[21] This reason for fiscal activism is additional to the reason provided in Section 3 above; it serves to reinforce that argument.

Figure 5 displays the simulations when the backward-looking price-setters respond to lagged output gap. we follow Steinsson (2003) to set η so that the weigh on lagged output is equal to the coefficient on the marginal cost in our Phillips curve when $\omega = 0$. This means that $\eta = \kappa$. It is clear that government spending now adjusts even after period 2. Government spending falls more in the initial periods compared with the responses when backward-looking price-setters do not respond to lagged output. As a result, output is lower, which not only helps to control inflation contemporaneously, but has an additional effect of helping to control inflation in the next period, due to the lagged output term in the Phillips curve.

[20]By contrast, when price-setters are purely forward-looking, as in the exercise shown up until now in this paper, inflation rises in the first period, but then drops below zero in period 2 and subsequent periods which helps the initial disinflation through inflation expectations (Currie and Levine, 1987, 1993; Woodford, 2003a). Given this difference, when inflation is persistent, the policymaker has to act more aggressively in order to control inflation, and so responses show larger fluctuation than the system without backward-looking price-setters.

[21]In fact the interactions between the forward-looking and backward-looking price-setters mean that fiscal and monetary policies influence inflation both intratemporally and intertemporally. The intratemporal effect makes fiscal policy more comparatively disadvantaged than monetary policy in controlling inflation relative to an economy in which $\eta = 0$. With this effect alone, it may be helpful to combine a fiscal expansion and a monetary contraction to control inflation. The intertemporal effect suggests that a fiscal contraction in the current period reduces both inflation and the period loss function in the next period. Luk and Vines (2015) analyse these effects in detail and conclude that the intertemporal effect dominates, which means that government spending falls more initially when $\eta > 0$ in response to a positive inflation shock.

The results of this subsection are summarised in the last two rows of Table 1.

5 Robustness II: Optimal Policy with different parameters

We showed that when the stock of government debt is high, it is possible that the optimal commitment policy is one in which the nominal interest rate falls in response to a positive inflation shock. In this section we discuss how robust these results are in the face of two particular alternative parameterisations.

We vary the persistence parameter of the inflation shock, ρ_μ. It is often assumed that the inflation shock persistence is high in order to match the high persistence in observed inflation data. Figure 6 presents the impulse responses to a 1 unit inflation shock in the full model (with endogenous capital and inflation persistence) with 60% debt-to-output ratio and with different persistence in the inflation shock process. It is clear that when the inflation shock is more persistent, it is more likely that the nominal interest rate behaves perversely in the first period, and that the government spending needs to fall by more in the first period. Given our benchmark calibration and with a 60% debt-to-output ratio, it is optimal to cut the nominal interest rate in the first period when the inflation shock persistence is beyond 0.4. The reason is simple: when the shock is more persistent, controlling inflation requires a prolonged and tighter monetary contraction which would lead to a larger accumulation of debt. As a result, there is a greater need to use loosening monetary policy and tighten fiscal policy initially to make sure that the stock of debt in the new steady-state is not too high.

Finally, we ask whether increasing the share of backward-looking price-setters has a similar effect to increasing the persistence of the inflation shock. It turns out that effect of increasing the share of backward-looking price-setters is somewhat more complex. Figure 7 shows the contours of the magnitude and direction of the initial jump in the nominal interest rate given different calibrations of the debt-to-output ratios and shares of backward-looking price-setters. We observe that when the steady-state stock of debt is high, and the share of backward-looking price-setters is either small or large, the initial nominal interest rate response is negative, but when the share of backward-looking price-setters is around 0.5, the interest rate response ceases to be negative. There are two effects underlying these findings. One effect is that as the share of backward-looking price-setters rises, expected future inflation has a smaller inflationary effect today, so the nominal interest rate is more likely to rise initially. The other effect is that when the share of backward-looking price-setter rises, controlling inflation is more difficult, and requires a more persistent rise in the nominal interest rate to control inflation, so the nominal interest rate is more likely to fall initially to offset the subsequent rise in debt. As the share of backward-looking price-setters goes from 0 to 1, the first effect dominates when the share of backward-looking price-setter is small, and the second effect dominates when the share of backward-looking price-setter is large; this is what gives rise to the non-linear effect.

6 Conclusion

In this paper, we have studied the optimal coordination of monetary and fiscal policy in a New Keynesian model when the policymaker acts with commitment. We first showed that – within a simple, standard, microfounded, new Keynesian framework – the 'conventional' view about fiscal policy is correct. This view states that it is undesirable to use fiscal policy as well as monetary to stabilise inflation, even when this is possible. Fiscal policy should be kept in the background and used only to ensure the solvency of the public sector.

In this model we have added endogenous public debt to the simple model. Policy now obviously requires some fiscal action if an explosion in public debt is to be avoided. We have explored how active fiscal policy needs to be during the process of moving to the new equilibrium. We have shown that in the longer term fiscal policy needs to be tightened to make room for the higher interest payments generated by the higher stock of public debt. When the stock of debt is high fiscal policy will need to be very contractionary in the short run - formally in the first period – and the nominal interest rate will actually need to fall, in order to control public debt in the best way possible. But we have also demonstrated that, even in this case, fiscal policy moves, in the second period, to the level required by fiscal solvency and plays no subsequent role in the active stabilisation of the economy.

We carried out two important generalisations of our model in order to check the robustness of our findings. If we add endogenous capital accumulation to the model we find that, in principle, having an active fiscal policy can be advantageous. Contracting fiscal policy after an inflation shock will moderate the reduction in the capital stock which happens during the adjustment process. We have shown that this can reduce the loss of welfare which occurs during a disinflation process since it will lessen the extent to which labour is forced to work inefficiently with an inadequate capital stock during that period of adjustment e adjustment. But we also showed that, given a realistic parameterisation of the model, the effects of this are small.

In addition we have shown that allowing price setters to be, at least in part, backward-looking has significant implications. When the backward-looking price-setters respond to lagged output, fiscal policy is tighter along the entire disinflation path. This is because, since backward-looking consumers do not increase their labour supply along the adjustment path, removing – for them – the comparative advantage that monetary policy has in the control of inflation.

It may turn out that it is inappropriate to make the key assumption about labour supply which underpins all of our findings, an assumption which lies at the centre of all the new Keynesian model. In particular it may be the case that the supply of labour is completely inelastic.[22] In that case monetary policy has no comparative advantage over fiscal policy in the control of inflation. In this case it *will* be the case that fiscal policy should share the burden of inflation control with monetary policy.[23]

Additional reasons as to why an active fiscal policy may be necessary may arise, if the policy problem is not control of inflation, but something else.

One case of this is if the requirement is to increase aggregate demand but the country has reached the zero bound in the setting of interest rates. If stimulus to aggregate demand is needed in these circumstances, then such a stimulus will need to come from somewhere else, rather than from cutting the interest rate. Quantitative easing - *i.e.* action to lower longer term interest rates - may be one such means. Fiscal expansion may be another way of achieving what is desired. Of course such an expansion must be coupled with a plan for ensuring that public debt is under longer-term control. This is a different problem from that discussed in this paper and we have deliberately not pursued

[22] There is a wide range of evidence on labour supply, much of which finds the supply of labour to be inelastic. For instance, see the review article by Chetty et al. (2011) and the references therein.

[23] It is straightforward to rework the analysis in Section 2 of the paper to show this; details are available from the authors on request.

it here.[24]

Similarly, if a country is a member of a monetary union then an active fiscal policy may be required if the country is subject to asymmetric shocks to aggregate demand. The first best outcome in such circumstances would be immediate adjustment of relative wages within the monetary union to redistribute demand within the union, away from the region in which it had increased (relative to the average) and towards the other region where the reverse had happened. But the effects of such redistribution of demand take time and there are also constraints on the relative movement of wages, particularly in a downward direction. In this case fiscal contraction may be needed in the region which is experiencing a positive demand shock compared with other part of the union, coupled with fiscal expansion in the other parts of the union. Of course, such changes in the relative fiscal positions within the union must be coupled with plan to ensure fiscal solvency in each part of the region, particularly in the part which is undergoing the fiscal expansion. Again, this is a different problem from that discussed in this paper and we have deliberately not pursued it here.

Finally the policy problem may be that of an excessive stock of public debt, as in advanced countries in the world at present, in the period after the global financial crisis. The policy problem in these circumstances is to work out the optimal speed at which this stock of debt should be reduced. In particular should there be a large fiscal consolidation so that public debt is reduced rapidly, or *vice versa*? This is a much bigger problem than that analysed in this paper. The solution will depend on whether, and for how long in the adjustment process, there is a zero bound in interest rates. Furthermore, the level to which it is desirable to reduce public debt will depend on features not examined here. In particular, the model which is necessary will need to have an overlapping-generations structure and will need to permit an analysis of the provision of pensions. All of this is a task for future work.

The many reasons as to why an active fiscal policy may be valuable suggest that it will be important to establish fiscal councils, of the kind which have been advocated by Calmfors and Wren-Lewis (2011). In these councils the examination of optimal fiscal policy, and the conduct of fiscal policy itself, could be better removed from partisan interest. The ambition is to better isolate fiscal policy from political influence, in the way

[24]Recent papers on the optimal coordination of fiscal and monetary policy in the presence of a zero bound to interest rates include Nakata 2013, Schmit 2013 and Burgert and Schmidt 2014.

which has happened with monetary policy.

Table 1: Response to a positive inflation shock under optimal commitment policy

	Low debt	High debt
Lump-sum tax	Government spending (g_t) does not move. The nominal interest rate (r_t) acts to stabilise inflation.	
Endogenous debt	g_t falls in period 1 and recovers slightly from period 2 onwards. g_t does not move from period 2 to the long run. In long run, $g_\infty < 0$. r_t rises in period 1, gradually falls back to 0.	g_t falls sharply in period 1 and recovers sharply from period 2 onwards. g_t does not move from period 2 to long run. In long run, $g_\infty < 0$. r_t *falls* in period 1. rises from period 2 onwards.
Endogenous Capital Accumulation	g_t falls in period 1 and recovers slightly from period 2 onwards. g_t moves slightly from period 2 to the long run. In long run, $g_\infty < 0$. r_t rises in period 1, gradually falls back to 0.	g_t falls sharply in period 1 and recovers sharply from period 2 onwards. g_t moves slightly from period 2 to long run. $g_\infty < 0$. r_t *falls* in period 1, rises from period 2 onwards.
Inflation persistence with no lagged output gap in the Phillips curve	g_t falls in period 1 and recovers slightly from period 2 onwards. g_t does not move from period 2 to the long run. In long run, $g_\infty < 0$. r_t rises in period 1, gradually falls back to 0.	g_t falls sharply in period 1 and recovers sharply from period 2 onwards. g_t does not move from period 2 to the long run. In long run, $g_\infty < 0$. r_t *falls* in period 1. rises from period 2 onwards.
Inflation persistence with lagged output gap in the Phillips curve	g_t falls in period 1 and recovers slightly from period 2 onwards. g_t remains lower than its long-run level for many periods after period 2. In long run, $g_\infty < 0$. r_t rises in period 1, gradually falls back to 0.	g_t falls sharply in period 1 and recovers sharply from period 2 onwards. g_t remains lower than its long-run level for many periods after period 2. In long run, $g_\infty < 0$. r_t *falls* in period 1. rises from period 2 onwards.

Table 2: Parameter values

Parameter	Simple NK Model with lump-sum tax tax	Model with endogenous debt	Model with endogenous capital accumulation	Model with inflation persistence with no lagged output gap in the PC	Model with inflation persistence with lagged output gap in the PC
β	0.99	0.99	0.99	0.99	0.99
σ	1	1	1	1	1
φ	0.5	0.5	0.5	0.5	0.5
ϵ	5	5	5	5	5
γ	0.75	0.75	0.75	0.75	0.75
θ_c	0.75	0.75	0.56	0.56	0.56
θ_g	0.25	0.25	0.2	0.2	0.2
α	0	0	0.34	0.34	0.34
δ	–	–	0.025	0.025	0.025
J''	–	–	-10	-10	-10
ω	0	0	0	0.75	0.75
η	–	–	–	0	$\kappa = \frac{(1-\beta)(1-\beta\gamma)}{\gamma}$
B	–	0.1, 2.4, 6	0.1, 2.4, 6	0.1, 2.4, 6	0.1, 2.4, 6
ρ_μ	0	0	0	0	0

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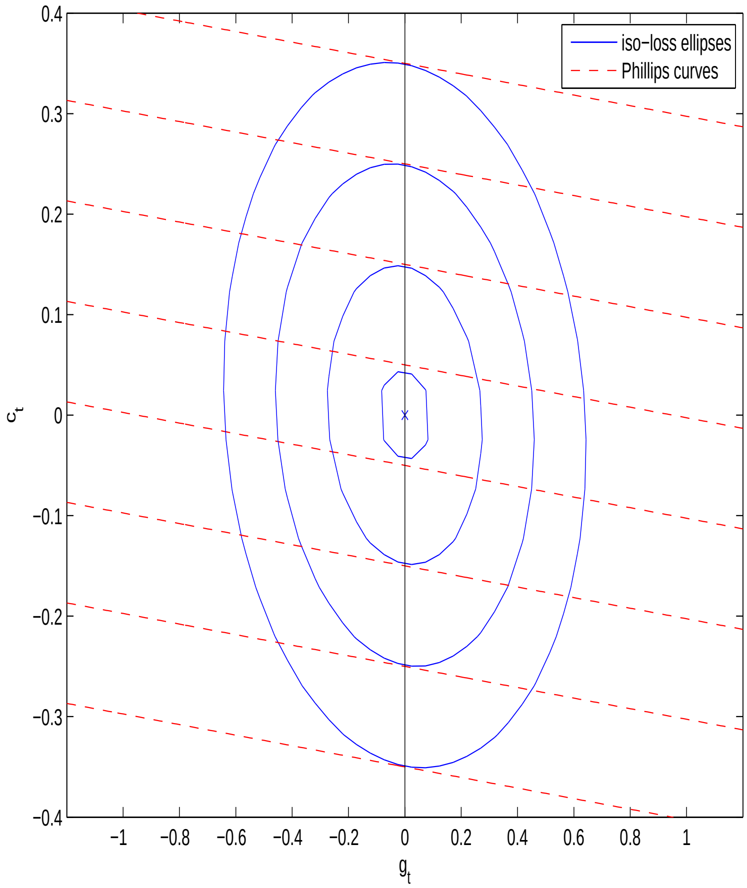
Figure 1: The optimal choice for consumption and government spending.



Note: The figure shows the iso-loss ellipse and the constant marginal cost curves in the simple model. This figure assumes $\varphi = 0.5, \theta = 0.75, \sigma = 2$.

A Appendix

A.1 Details of the general New-Keynesian Model

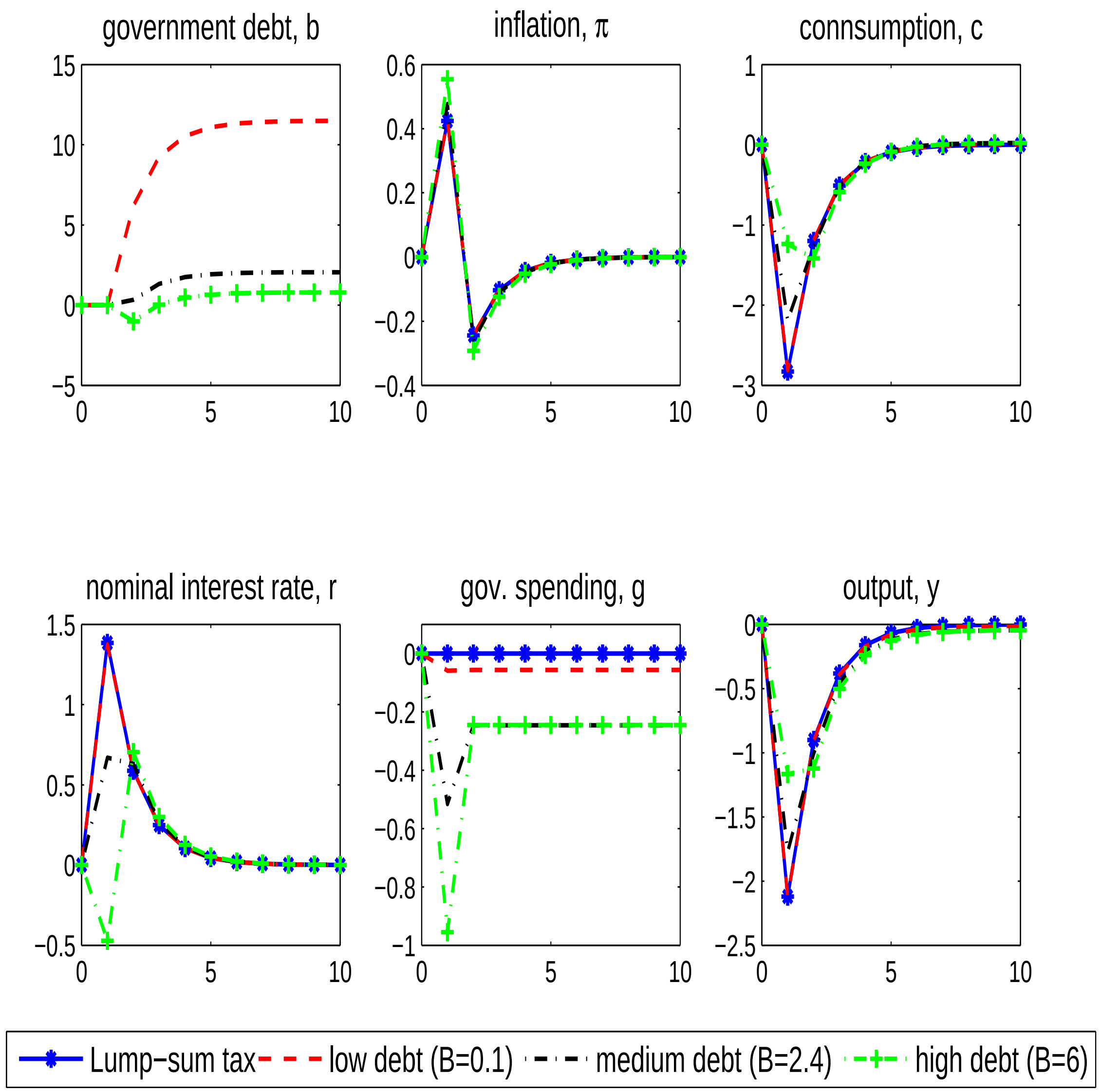
This appendix derives the details of the general New-Keynesian model, with endogenous public debt, endogenous capital accumulation and inflation persistence.

A.1.1 Consumers

Consumers are homogeneous. They have the following utility:

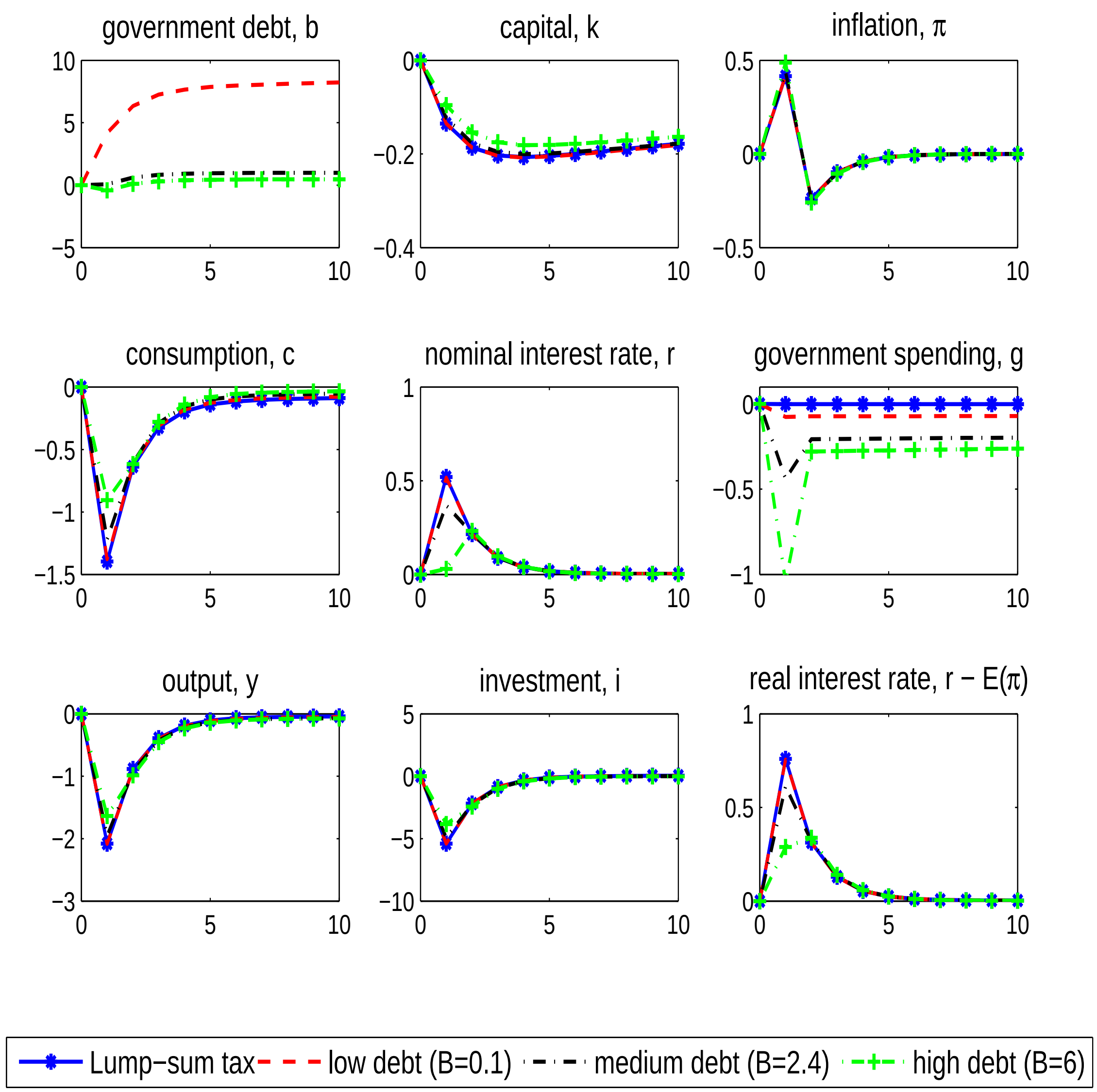
$$U_t = E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\frac{C_s^{1-\sigma}}{1-\sigma} + \chi \frac{G_s^{1-\sigma}}{1-\sigma} - \frac{N_s^{1+\varphi}}{1+\varphi} \right), \tag{49}$$

Figure 2: Impulse response to a 1% positive inflation shock under optimal commitment policy in the simple NK model without capital accumulation or inflation persistence.



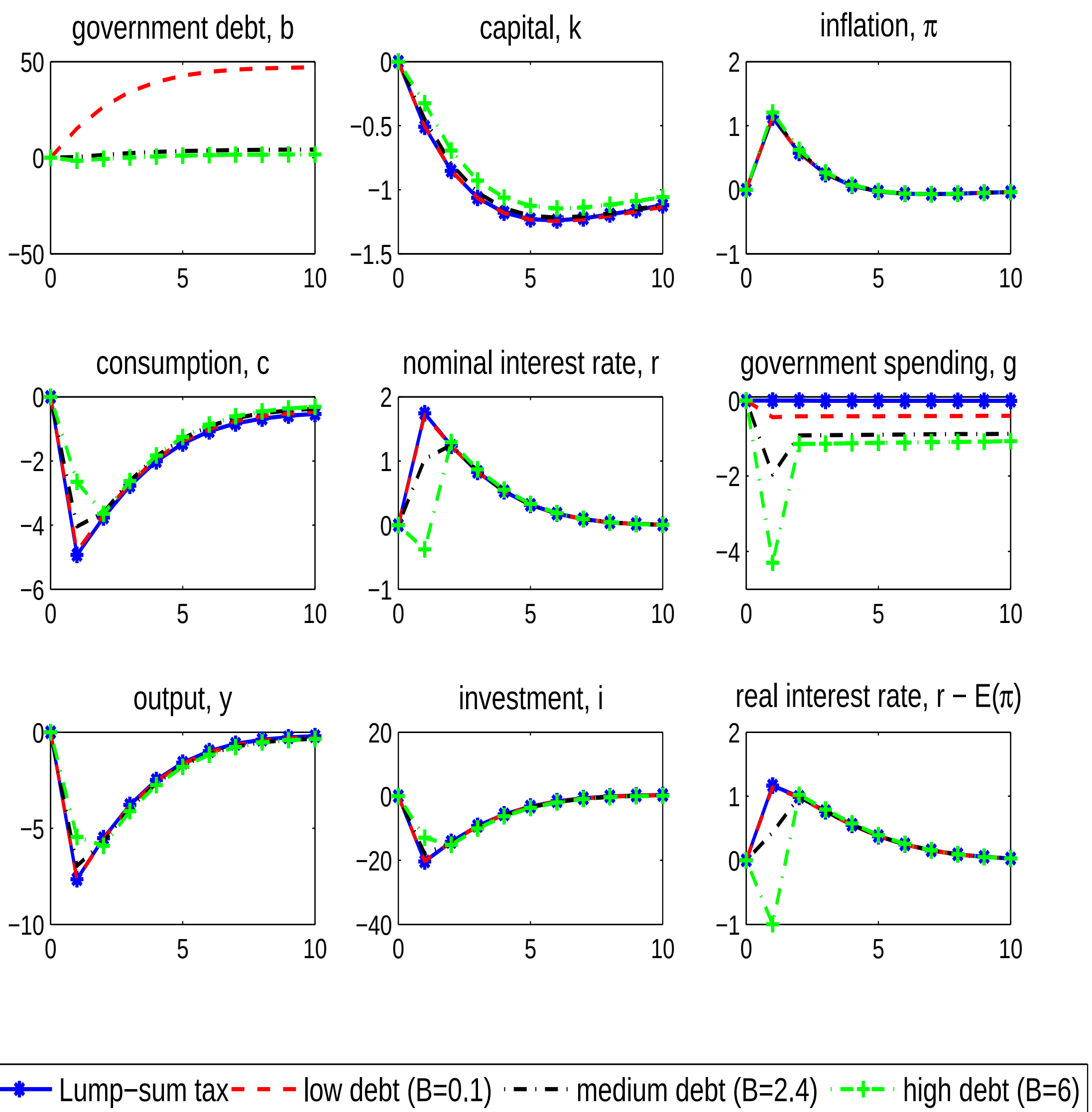
Note: Blue solid lines show the response in the full model in which the government has access to lump-sum tax and runs a balanced budget. Red dashed lines show the response in the model in which debt evolves endogenously with steady-state debt-to-output ratio $B/Y = 0.1$. Black dash-dotted lines the response in the model in which debt evolves endogenously with steady-state debt-to-output ratio $B/Y = 2.4$. Green dash-dotted lines the response in the model in which debt evolves endogenously with steady-state debt-to-output ratio $B/Y = 6$. Other parameters are: $\beta = 0.99, \varphi = 0.5, \epsilon = 5, \gamma = 0.75, \theta = 0.75$.

Figure 3: Impulse response to a 1% positive inflation shock under optimal commitment policy in the model with endogenous public debt and capital accumulation.



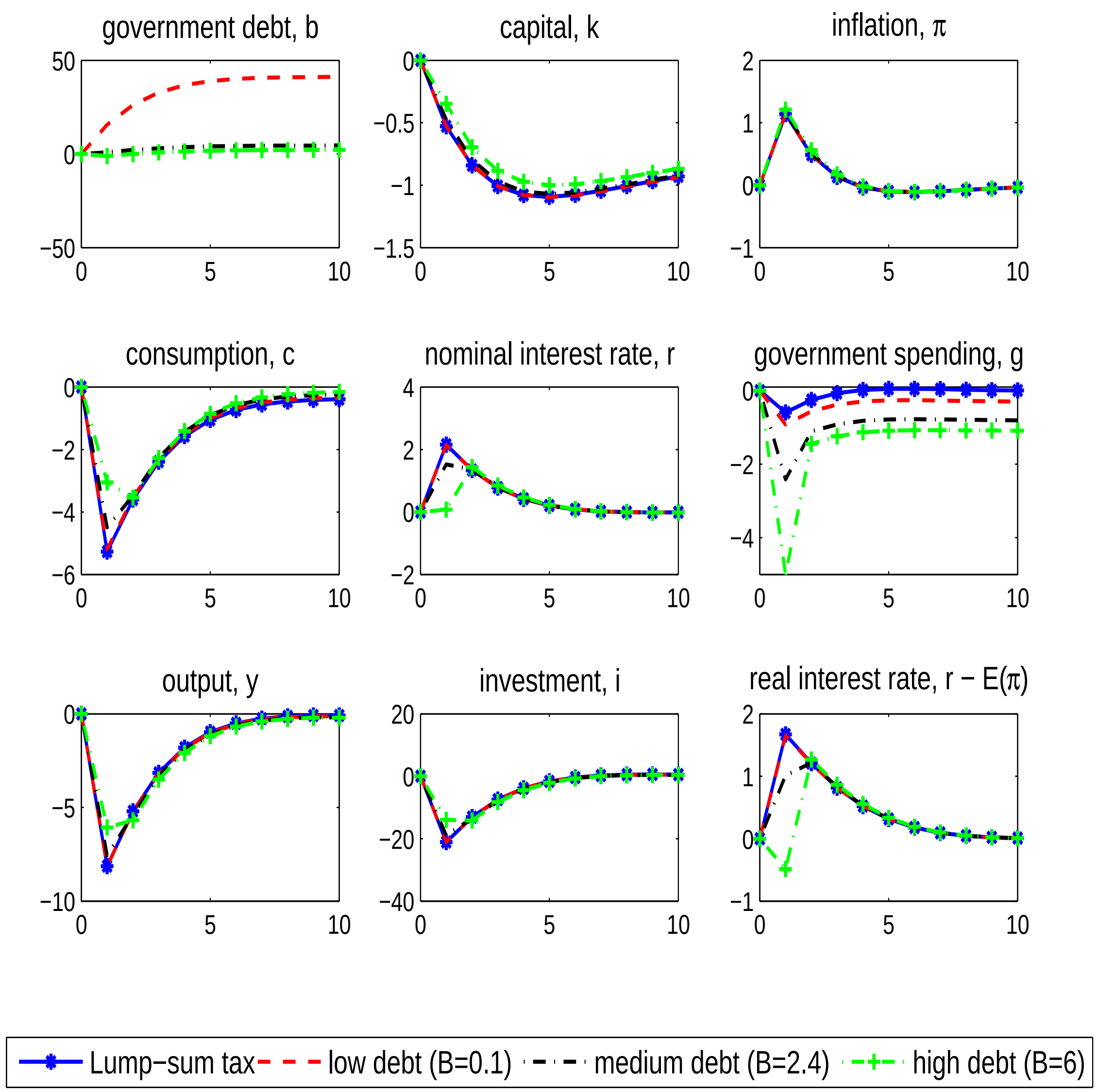
Note: Blue solid lines show the response in the full model in which the government has access to lump-sum tax and runs a balanced budget. Red dashed lines show the response in the model in which debt evolves endogenously with steady-state debt-to-output ratio $B/Y = 0.1$. Black dash-dotted lines the response in the model in which debt evolves endogenously with steady-state debt-to-output ratio $B/Y = 2.4$. Green dash-dotted lines the response in the model in which debt evolves endogenously with steady-state debt-to-output ratio $B/Y = 6$. We set the share of backward-looking price-setters to $\omega = 0$. Other model parameter values are given in Table 2.

Figure 4: Impulse response to a 1% positive inflation shock under optimal commitment policy in the model with endogenous public debt, capital accumulation and backward-looking price-setters who do not respond to lagged output gap.



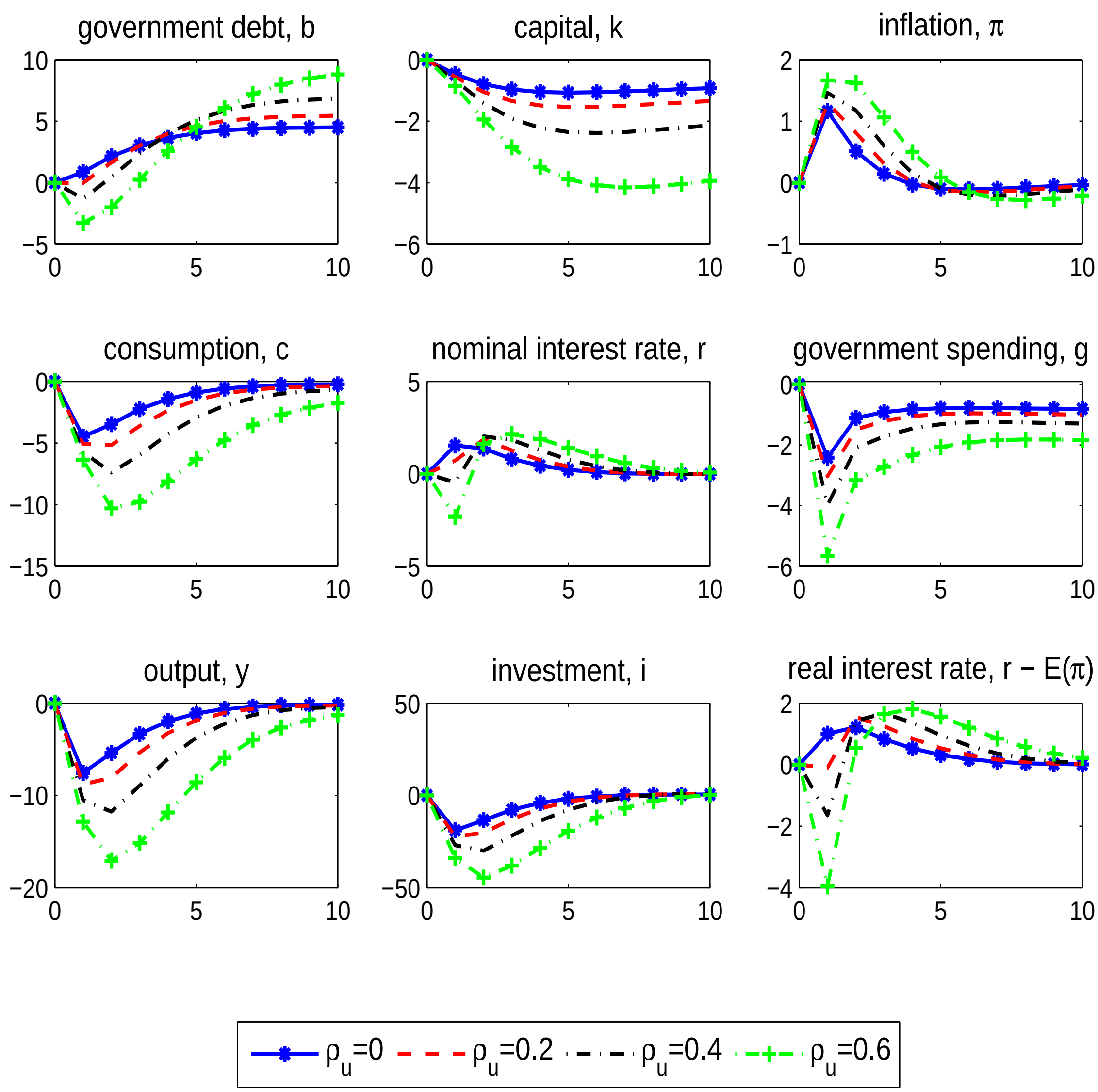
Note: Blue solid lines show the response in the full model in which the government has access to lump-sum tax and runs a balanced budget. Red dashed lines show the response in the model in which debt evolves endogenously with steady-state debt-to-output ratio $B/Y = 0.1$. Black dash-dotted lines the response in the model in which debt evolves endogenously with steady-state debt-to-output ratio $B/Y = 2.4$. Green dash-dotted lines the response in the model in which debt evolves endogenously with steady-state debt-to-output ratio $B/Y = 6$. We set the backward-looking price-setters response to lagged output gap to $\eta = 0$. Other model parameter values are given in Table 2.

Figure 5: Impulse response to a 1% positive inflation shock under optimal commitment policy in the model with endogenous public debt, capital accumulation and backward-looking price-setters who respond to lagged output gap.



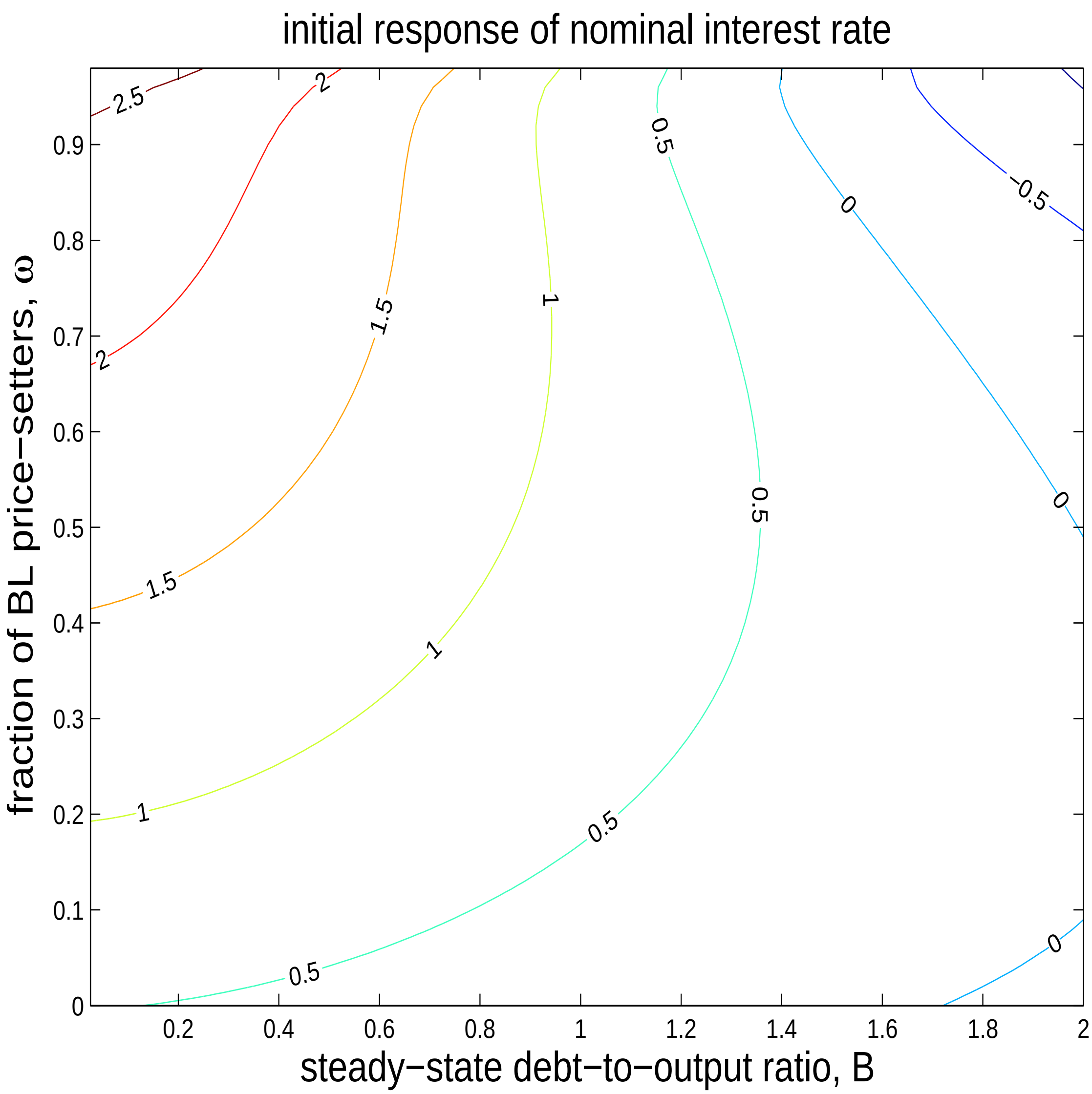
Note: Blue solid lines show the response in the full model in which the government has access to lump-sum tax and runs a balanced budget. Red dashed lines show the response in the model in which debt evolves endogenously with steady-state debt-to-output ratio $B/Y = 0.1$. Black dash-dotted lines the response in the model in which debt evolves endogenously with steady-state debt-to-output ratio $B/Y = 2.4$. Green dash-dotted lines the response in the model in which debt evolves endogenously with steady-state debt-to-output ratio $B/Y = 6$. Other model parameter values are given in Table 2.

Figure 6: Impulse response to a 1% positive inflation shock under optimal commitment policy in the full model with different inflation shock persistence.



We use the steady-state debt-to-output ratio $B/Y = 2.4$. Other model parameter values are given in Table 2.

Figure 7: Contours of the initial response of the nominal interest rate under the optimal commitment policy when a 1% positive inflation shock hits the economy.



Note: In these figures, we use benchmark parameter values given in Table 2.

where C_t denotes consumption in period t, G_t denotes government expenditure and N_t the labour supply. E_t is the expectation operator conditional on the information at time t. Consumers supply labour, consume, save by buying government bonds and make capital investments. The budget constraint of the representative consumer is as follows:

$$P_t I_t + P_t C_t + E_t(Q_{t,t+1} D_{t+1}) = D_t + (1-\tau)(W_t N_t + R_t^K K_t + \Omega_t) - T_t, \tag{50}$$

where K_t denotes the aggregate capital stock and I_t denotes investment. R_t^K denotes the return on capital through renting the capital to the firms. P_t and W_t denote the price level and the nominal wage respectively. We assume τ is an exogenous income tax rate. Ω_t is the profit from the firms and T_t is a lump-sum tax to finance a production subsidy which offsets distortions caused by monopolistic competition. $Q_{t,t+1}$ is the stochastic discount factor which determines the price to the consumer to transfer a state-contingent amount D_{t+1} of wealth from period t to $t+1$. Its relation to the nominal interest rate is $E_t(Q_{t,t+1}) = R_t^{-1}$. The usual transversality condition applies.

Capital investment is subject to an adjustment cost. The capital accumulation equation is:

$$K_{t+1} = (1-\delta)K_t + J\left(\frac{I_t}{K_t}\right) K_t, \tag{51}$$

where δ is the capital depreciation rate. The formulation of this capital adjustment cost is the same as Hayashi (1982). The function J is assumed to be a quadratic function which satisfies $J(\delta) = \delta, J'(\delta) = 1$ and J'' is a negative constant.

Consumers maximise utility subject to the budget constraint and the capital accumulation equation. The first order conditions are as follows:

$$1 = \beta R_t E_t\left(\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+1}}\right), \tag{52}$$

$$C_t^\sigma N_t^\varphi = (1-\tau)\frac{W_t}{P_t}, \tag{53}$$

$$E_t\left(R_t \frac{P_t}{P_{t+1}}\right) = J'\left(\frac{I_t}{K_t}\right) E_t\left[\frac{(1-\tau)R_{t+1}^K}{P_{t+1}} + \frac{1-\delta + J\left(\frac{I_{t+1}}{K_{t+1}}\right) - \frac{I_{t+1}}{K_{t+1}} J'\left(\frac{I_{t+1}}{K_{t+1}}\right)}{J'\left(\frac{I_{t+1}}{K_{t+1}}\right)}\right] \tag{54}$$

The optimal decision for investment equates the marginal costs and benefits of acquiring an additional unit of capital. The marginal cost of investment is the forgone consumption in the current period. The marginal benefits are the value of the after-tax rental income in the next period, plus the value of next period's capital which accounts for depreciation and the change in capital adjustment costs.

A.1.2 Final Goods Firm

On the production side, there is a final goods firm which mixes intermediate goods using a Dixit-Stiglitz (1977) aggregator such that $Y_t = \left(\int_0^1 Y_t(i)^{\frac{\epsilon_t - 1}{\epsilon_t}} di\right)^{\frac{\epsilon_t}{\epsilon_t - 1}}$ and the demand for each type of good is given by $Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon_t} Y_t$, where $P_t \equiv \left[\int_0^1 P_t(i)^{1-\epsilon_t} di\right]^{\frac{1}{1-\epsilon_t}}$.

A.1.3 Intermediate Goods Firm

Intermediate goods firms are assumed to be monopolistically competitive. Each firm i produces with capital and labour. The production function for firm i is

$$Y_t(i) = A_t N_t(i)^{1-\alpha} K_t(i)^{\alpha}. \tag{55}$$

where the productivity A_t is an exogenous AR(1) process such that $\hat{A}_{t+1} = \rho_A \hat{A}_t + \xi_{A,t+1}$. Aggregate labour is defined as $N_t \equiv \int_0^1 N_t(i) di$.

In each period, each intermediate goods firm chooses labour and capital input to maximise profit. The profit of firm i in period t is:

$$\Omega_t(i) = P_t(i) Y_t(i) - \frac{1}{\mu_w}(W_t N_t(i) + R_t^K K_t(i)). \tag{56}$$

Firms receive a subsidy $\mu_w = \epsilon/(\epsilon - 1)/(1 - \tau)$ from the government which ensures that the steady state is efficient. The subsidy is paid for by a lump sum tax, T_t. We denote the Lagrange multiplier to the production function as $P_t MC_t(i)/\mu_w$, where $MC_t(i)$ also has the interpretation as the real marginal cost of firm i. The demands for labour and capital are:

$$N_t(i) = (1-\alpha) MC_t(i) \frac{Y_t(i)}{W_t/P_t}, \tag{57}$$

$$K_t(i) = \alpha MC_t(i) \frac{Y_t(i)}{R_t^K/P_t}. \tag{58}$$

Since the factor prices are equal across firms, the capital-to-labour ratio is identical across firms because:

$$\frac{N_t(i)}{K_t(i)} = \frac{(1-\alpha) R_t^K/P_t}{\alpha W_t/P_t} = \frac{N_t}{K_t}, \tag{59}$$

where $K_t \equiv \int_0^1 K_t(i) di$.

The marginal cost is also identical across firms, such that:

$$MC_t(i) = \frac{\left(\frac{W_t}{P_t}\right)^{1-\alpha} \left(\frac{R_t^K}{P_t}\right)^{\alpha}}{A_t (1-\alpha)^{1-\alpha} \alpha^{\alpha}} \equiv MC_t. \tag{60}$$

The aggregate production function is

$$Y_t = \frac{A_t K_t^{\alpha} N_t^{1-\alpha}}{\Delta_t}, \tag{61}$$

where $\Delta_t \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon_t} di$ is the price dispersion.

A.1.4 Price Setting

Price setting is modeled as a mix of Calvo (1983) contracting and rule-of-thumb behaviour. Firms recalculate their prices with fixed probability $(1-\gamma)$. If prices are recalculated then a proportion ω of the price resetting agents use a rule of thumb to set their price and proportion $(1-\omega)$ calculate the optimum price. With probability γ prices are not recalculated and are assumed to rise at the average rate of inflation.

We assume backward-looking agents set their prices P_t^{BL} using the rule of thumb:

$$P_t^{BL} = P_{t-1}^{*} \Pi_{t-1} \left(\frac{Y_{t-1}}{Y_{t-1}^{n}}\right)^{\eta}, \tag{62}$$

where $\Pi_t = P_t/P_{t-1}$ and Y_t^n is the flexible-price equilibrium of output. We use superscript * to denote the average price reset by the firms. This average price reset by the firms is a weighted average between forward-looking (P_t^{FL}) and backward-looking prices (P_t^{BL}):

$$P_t^{*} = (P_t^{FL})^{1-\omega} (P_t^{BL})^{\omega}, \qquad 0 \leq \omega < 1. \tag{63}$$

This rule of thumb is identical to the formulation in Steinsson (2003), and when η is set to 0 the rule reduces to the formulation in Gali and Gertler (1999). η defines the relative weight of output considerations in the rule of thumb.

The forward-looking price-setters solve the first order conditions for profit maximisation and obtain the optimal solution. The Lagrangian for the price setting problem is:

$$\begin{aligned} Lagr_t \;\; = \;\; & E_t \sum_{s=0}^{\infty} \gamma^s Q_{t,t+s} \Bigg(P_t(i) Y_{t+s}(i) - \frac{1}{\mu_w} \left(W_{t+s} N_{t+s}(i) + R_{t+s}^{K} K_{t+s}(i) \right) \\ & + \frac{P_{t+s} MC_{t+s}}{\mu_w} \left(A_{t+s} N_{t+s}(i)^{1-\alpha} K_{t+s}(i)^{\alpha} - P_t(i) Y_{t+s}(i) \right) \Bigg). \end{aligned} \tag{64}$$

Forward-looking price-setters choose the same price P_t^{FL} to satisfy the following first order condition:

$$E_t \sum_{s=0}^{\infty} \gamma^s Q_{t,t+s} \left(\frac{P_t^{FL}}{P_{t+s}}\right)^{-1-\epsilon_{t+s}} Y_{t+s} \left((\epsilon_{t+s} - 1) \frac{P_t^{FL}}{P_{t+s}} - \epsilon_{t+s} \frac{MC_{t+s}}{\mu_w} \right) = 0. \tag{65}$$

The evolution of the aggregate price is as follows:

$$P_t = \left(\gamma P_{t-1}^{1-\epsilon_t} + (1-\gamma)\omega\left(P_t^{BL}\right)^{1-\epsilon_t} + (1-\gamma)(1-\omega)\left(P_t^{FL}\right)^{1-\epsilon_t}\right)^{\frac{1}{1-\epsilon_t}}. \tag{66}$$

The price dispersion, Δ_t, evolves as follows:

$$\Delta_t = \gamma \Pi_t^{\epsilon_t}\Delta_{t-1} + (1-\gamma)\left[\omega\left(\frac{P_t^{BL}}{P_t}\right)^{-\epsilon_t} + (1-\omega)\left(\frac{P_t^{FL}}{P_t}\right)^{-\epsilon_t}\right]. \tag{67}$$

where $\Pi_t = P_t/P_{t-1}$.

A.1.5 The Government and Goods Market Clearing

The government buys goods, taxes income with a constant tax rate τ and issues a nominal debt $\bar{B}$. The evolution of the nominal debt is given by:

$$B_{t+1} = R_t\left(B_t\frac{P_{t-1}}{P_t} + G_t - \tau Y_t\right). \tag{68}$$

where $B_t = \bar{B}_t/P_{t-1}$. The goods market clearing condition is given by:

$$Y_t = C_t + I_t + G_t. \tag{69}$$

In the steady state assume $C = \theta_c Y$, $G = \theta_g Y$ and $I = \theta_i Y$, where $\theta_c, \theta_g, \theta_i$ are the shares of private consumption, government spending and investment in output.

This completes the description of the model. In sum, the model consists of 14 equations (51), (52), (53), (54), (59), (60), (61), (62), (63), (65), (66), (67), (68) and (69). Together with the policy rules of the nominal interest rate and government spending, this system of equations enables us to solve for the following endogenous variables:

$$\{K_t, C_t, N_t, I_t, \frac{W_t}{P_t}, \frac{R_t}{P_t}, MC_t, Y_t, \Pi_t, \frac{P_t^{BL}}{P_t}, \frac{P_t^{FL}}{P_t}, \frac{P_t^*}{P_t}, B_t, \Delta_t, R_t, G_t\}.$$

A.1.6 The Case of Flexible Price and the System in 'Gap' Form

In the case of flexible price, the social planner maximises utility, Equation (49), subject to the goods market clearing condition, the evolution of capital and the production function.

The first order conditions are:

$$\left(C_t^n\right)^{-\sigma} = \chi \left(G_t^n\right)^{-\sigma}, \tag{70}$$

$$\left(C_t^n\right)^{\sigma}\left(N_t^n\right)^{\varphi} = (1-\alpha)\frac{Y_t^n}{N_t^n}, \tag{71}$$

$$E_t\frac{\left(C_{t+1}^n\right)^{\sigma}}{\beta\left(C_t^n\right)^{\sigma}} = J'\left(\frac{I_t^n}{K_t^n}\right)E_t\left[\alpha\frac{Y_{t+1}^n}{K_{t+1}^n} + \frac{1-\delta+J\left(\frac{I_{t+1}^n}{K_{t+1}^n}\right)-\frac{I_{t+1}^n}{K_{t+1}^n}J'\left(\frac{I_{t+1}^n}{K_{t+1}^n}\right)}{J'\left(\frac{I_{t+1}^n}{K_{t+1}^n}\right)}\right]. \tag{72}$$

where the superscript 'n' denotes the efficient flexible price level. We follow Neiss and Nelson (2003) and define the natural level of output such that the natural level of output in a given period depends on the capital stock consistent with the flexible-price model. Edge (2003) show that with this assumption, the second-order approximated utility-based welfare function can be completely characterised in terms of the 'gap' variables.

The first order conditions, (70) - (72), together with the constraints of the economy, namely the evolution of the capital stock (51), the goods market clearing condition (69) and the production function (61) with price dispersions $\Delta = 1$ form a system which, given the initial values of the state variables K_0^n, A_0 and the exogenous productivity shock process, describes the dynamic behaviour of $\{K_{t+1}^n, C_t^n, N_t^n, I_t^n, Y_t^n, G_t^n\}$.

The flexible price system can be log-linearised and solved using standard methods to yield the transition paths of the macroeconomic variables in the following form:

$$\hat{K}_{t+1}^n = \psi_K\hat{K}_t^n + \zeta_K\hat{A}_t,$$
$$\hat{X}_t^n = \psi_X\hat{K}_t^n + \zeta_X\hat{A}_t,$$

where $X = \{C,\ N,\ I,\ Y,\ G\}$. The natural interest rate $\hat{R}_t^n$ is given by $\hat{R}_t^n = \sigma\hat{C}_t^n - \sigma E_t\hat{C}_{t+1}^n$.

The model with nominal rigidities can be written in terms of the deviations from the natural levels as follows (Define $x_t \equiv \hat{X}_t - \hat{X}_t^n$):

$$c_t = \sigma E_t c_{t+1} - \sigma^{-1}\left(r_t - E_t\pi_{t+1}\right), \tag{73}$$

$$E_t(\Delta k_{t+1}) = \beta E_t(\Delta k_{t+2}) + \frac{1}{J''}\left(r_t - E_t\pi_{t+1}\right) - \frac{(1-(1-\delta)\beta)}{J''}E_t\left(\frac{1+\varphi}{1-\alpha}y_{t+1} + \sigma c_{t+1} - \frac{1+\varphi\alpha}{1-\alpha}k_{t+1}\right) \tag{74}$$

$$\pi_t = \chi^{FL}\beta E_t\pi_{t+1} + \chi^{BL}\pi_{t-1} + \kappa_c c_t + \kappa_{y0}y_t + \kappa_{y1}y_{t-1} - \kappa_k k_t + \kappa_c\hat{\mu}_t, \tag{75}$$

$$y_t = \theta_c c_t + \theta_i\left(\frac{1}{\delta}k_{t+1} - \frac{(1-\delta)}{\delta}k_t\right) + \theta_g g_t, \tag{76}$$

$$\hat{B}_{t+1} = r_t + \frac{1}{\beta}\left(\hat{B}_t - \pi_t + \frac{\theta_g}{B}g_t - \frac{\tau}{B}y_t\right) + \hat{\nu}_t. \tag{77}$$

where $B_t \equiv \bar{B}_t/P_{t-1}$ denotes the real stock of debt, and B is the steady-state debt to output ratio, and $\hat{\nu}_t = \left(\sigma\psi_C(1-\psi_K) + \frac{\theta_g}{\beta B}\psi_C - \frac{\tau}{\beta B}\psi_Y\right)\hat{K}_t^n + \left(\sigma\left(\zeta_C(1-\rho_A) - \psi_C\zeta_K\right) + \frac{\theta_g}{\beta B}\zeta_C - \frac{\tau}{\beta B}\zeta_Y\right)\hat{A}_t$. $\mu_t = \epsilon_t/(\epsilon_t - 1)$, and we assume an AR(1) process for the cost-push shock, *i.e.* $\hat{\mu}_{t+1} = \rho_\mu\hat{\mu}_t + \xi_{\mu,t+1}$.

A welfare-based loss function for the policymaker can be derived using a second-order approximation of the utility function around the efficient steady state. The loss function is

$$L_t = -\frac{C^{1-\sigma}}{2\theta_c}E_t\sum_{s=t}^{\infty}\beta^{s-t}l_s + t.i.p + O(3), \tag{78}$$

where

$$\begin{aligned} l_s = {} & \sigma\theta_c c_s^2 + \sigma\theta_g g_s^2 + \varphi(1-\alpha)n_s^2 + \alpha(1-\alpha)(n_s - k_s)^2 - \frac{\theta_i}{\delta}J''\left(\Delta k_{s+1}\right)^2 \\ & \underbrace{+\frac{\epsilon}{\kappa}\pi_s^2 + 2\frac{\epsilon}{\kappa}\phi_1\left(\Delta\pi_s\right)y_{s-1} + \frac{\epsilon}{\kappa}\phi_2\left(\Delta\pi_s\right)^2 + \frac{\epsilon}{\kappa}\phi_3 y_{s-1}^2}_{Var_i(\hat{P}_t(i))}, \end{aligned} \tag{79}$$

and

$$\kappa = \frac{(1-\gamma\beta)(1-\gamma)}{\gamma}, \qquad \phi_1 = -\frac{\omega\eta(1-\gamma)}{\gamma(1-\omega)}, \qquad \phi_2 = \frac{\omega}{\gamma(1-\omega)}, \qquad \phi_3 = \frac{(1-\gamma)^2\omega\eta^2}{\gamma(1-\omega)}.$$

The next two subsections detail the derivations of the hybrid Phillips curve and the social loss function.

A.2 Derivation of the Hybrid Phillips Curve in the Model with Capital

We define the following relative price variables:

$$p_t^{FL} = \frac{P_t^{FL}}{P_t}, \qquad p_t^{BL} = \frac{P_t^{BL}}{P_t}, \qquad p_t^* = \frac{P_t^*}{P_t}.$$

Equation (63) can be written as:

$$\hat{p}_t^* = (1-\omega)\hat{p}_t^{FL} + \omega\hat{p}_t^{BL}. \tag{80}$$

Equation (62) can be written as:

$$\hat{p}_t^{BL} = (1-\omega)\hat{p}_{t-1}^{FL} + \omega\hat{p}_{t-1}^{BL} - \pi_t + \pi_{t-1} + \eta y_{t-1}, \tag{81}$$

The evolution of price, Equation (66), is log-linearised as:

$$\pi_t = \frac{\omega(1-\gamma)}{(\gamma} p_t^{BL} - \frac{(1-\omega)(1-\gamma)}{\gamma} \hat{p}_t^{FL}. \quad (82)$$

Using these three equations to eliminate $\hat{p}_t^{BL}$ and $\hat{p}_t^*$, we obtain:

The forward-looking price-setters will set their prices to maximise their expected discounted future profits. The first order condition is:

$$E_t \sum_{s=0}^{\infty} \gamma^s Q_{t,t+s} \left(\frac{P_t^{FL}}{P_{t+s}} \right)^{-1-\epsilon_{t+s}} Y_{t+s} \left((\epsilon_{t+s} - 1) \frac{P_t^{FL}}{P_{t+s}} - \epsilon_{t+s} \frac{MC_{t+s}}{\mu_w} \right) = 0. \quad (83)$$

We define $\mu_t \equiv \epsilon_t/(\epsilon_t - 1)$. When prices are fully flexible and there is no fluctuation in monopolistic power, the above first order condition becomes $\mu MC_t = \mu_w$. Using the labour demand equation, Equation (57), and the labour supply equation, Equation (53), we can write the real marginal cost as:

$$MC_t = \frac{N_t \times \frac{W_t}{P_t}}{(1-\alpha)\Delta_t Y_t} = \frac{N_t^{1+\varphi} C_t^{\sigma}}{(1-\tau)(1-\alpha)\Delta_t Y_t}. \quad (84)$$

From Equation (71) we know that when prices are flexible, $N_t^{1+\varphi} C_t^{\sigma} = (1-\alpha)Y_t$. Therefore, when the subsidy μ_w satisfies $\mu/(1-\tau) = \mu_w$, the sticky-price economy will have the same steady state as the efficient flexible-price steady state.

The terms in the brackets of the first order condition, Equation (83), is log-linearised as follows:

$$\frac{P_t^{FL}}{P_{t+s}} - \mu_{t+s} MC_{t+s} = \hat{P}_t^{FL} - \hat{P}_{t+s} - \hat{MC}_{t+s} - \hat{\mu}_{t+s} = \hat{p}_t^{FL} - \sum_{k=1}^{s} \pi_{t+k} - \hat{MC}_{t+s} - \hat{\mu}_{t+s}.$$

Hence, the first order condition, Equation (83), is log-linearised as follows:

$$\begin{aligned} 0 &= E_t \sum_{s=t}^{\infty} (\gamma\beta)^{s-t} \left(\hat{p}_t^{FL} - \sum_{k=1}^{s-t} \pi_{t+k} - \hat{MC}_s - \hat{\mu}_s \right), \\ \hat{p}_t^{FL} &= (1-\gamma\beta) E_t \sum_{s=t}^{\infty} (\gamma\beta)^{s-t} \sum_{k=1}^{s-t} \pi_{t+k} + (1-\gamma\beta) E_t \sum_{s=t}^{\infty} (\gamma\beta)^{s-t} \left(\hat{MC}_s + \hat{\mu}_s \right), \\ \hat{p}_t^{FL} &= E_t \sum_{s=1}^{\infty} (\gamma\beta)^s \pi_{t+s} + (1-\gamma\beta) E_t \sum_{s=t}^{\infty} (\gamma\beta)^{s-t} \left(\sigma\hat{C}_s + (1+\varphi)\hat{N}_s - \hat{Y}_s + \hat{\mu}_s \right). \quad (85) \end{aligned}$$

The first order condition in quasi-difference form can be written as follows:

$$\hat{p}_t^{FL} = \gamma\beta E_t \hat{p}_{t+1}^{FL} + \gamma\beta E_t \pi_{t+1} + (1-\gamma\beta) \left(\sigma\hat{C}_t + (1+\varphi)\hat{N}_t - \hat{Y}_t + \hat{\mu}_t \right). \quad (86)$$

To continue to derive the Phillips curve, first write the above equation in the form of deviations from the efficient steady state as follows:

$$\hat{p}_t^{FL} = \gamma\beta E_t \hat{p}_{t+1}^{FL} + \gamma\beta E_t \pi_{t+1} + (1-\gamma\beta)\left(\sigma c_t + (1+\varphi)n_t - y_t + \hat{\mu}_t\right). \tag{87}$$

We also eliminate labour using the production function:

$$\hat{p}_t^{FL} = \gamma\beta E_t \hat{p}_{t+1}^{FL} + \gamma\beta E_t \pi_{t+1} + (1-\gamma\beta)\left(\sigma c_t + \frac{\alpha+\varphi}{1-\alpha} y_t - \frac{\alpha(1+\varphi)}{1-\alpha} k_t + \hat{\mu}_t\right). \tag{88}$$

We combine (81), (82) and (88) to obtain the Phillips curve:

$$\pi_t = \chi^{FL}\beta E_t \pi_{t+1} + \chi^{BL}\pi_{t-1} + \kappa_c c_t + \kappa_{y0} y_t + \kappa_{y1} y_{t-1} - \kappa_k k_t + \kappa_\mu \hat{\mu}_t, \tag{89}$$

where

$$\begin{aligned}
\chi^{FL} &= \frac{\gamma}{\omega(1-\gamma+\gamma\beta)+\gamma}, \qquad \chi^{BL} = \frac{\omega}{\omega(1-\gamma+\gamma\beta)+\gamma}, \\
\kappa_c &= \frac{\sigma(1-\gamma)(1-\omega)(1-\gamma\beta)}{\omega(1-\gamma+\gamma\beta)+\gamma}, \qquad \kappa_{y0} = \frac{1}{\sigma}\frac{\alpha+\varphi}{1-\alpha}\kappa_c - \frac{\gamma\beta\omega\eta(1-\gamma)}{\omega(1-\gamma+\gamma\beta)+\gamma}, \\
\kappa_{y1} &= \frac{\omega\eta(1-\gamma)}{\omega(1-\gamma+\gamma\beta)+\gamma}, \qquad \kappa_k = \frac{\alpha(1+\varphi)}{\sigma(1-\alpha)}\kappa_c, \qquad \kappa_\mu = \frac{1}{\sigma}\kappa_c.
\end{aligned}$$

When there is no backward-looking price-setters so that $\omega = 0$, and there is no capital so that $\alpha = 0$. The Phillips curve reduces to:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa(\sigma c_t + \varphi y_t + \hat{\mu}_t).$$

where $\kappa = (1-\gamma)(1-\gamma\beta)/\gamma$.

A.3 Second-order Approximation of the Social Welfare Function

The utility function is:

$$U_t = E_t \sum_{s=t}^{\infty} \beta^{s-t}\left(\frac{C_s^{1-\sigma}}{1-\sigma} + \chi\frac{G_s^{1-\sigma}}{1-\sigma} - \frac{N_s^{1+\varphi}}{1+\varphi}\right). \tag{90}$$

The second-order approximation of consumption utility is :

$$\frac{C_s^{1-\sigma}}{1-\sigma} = \frac{C^{1-\sigma}}{1-\sigma} + C^{1-\sigma}\left(\hat{C}_s + \frac{1}{2}(1-\sigma)\hat{C}_s^2\right) + O(3). \tag{91}$$

A similar expression can be derived for the second-order approximation of the utility from government spending.

The disutility of labour can be approximated as follows:

$$\frac{N_t^{1+\varphi}}{1+\varphi} = \frac{N^{1+\varphi}}{1+\varphi} + N^{1+\varphi}\left(\hat{N}_t + \frac{1}{2}(1+\varphi)\hat{N}_t^2\right) + O(3). \tag{92}$$

Therefore,

$$\begin{aligned} U_t = {} & E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(C^{1-\sigma}\left(\hat{C}_s + \frac{1}{2}(1-\sigma)\hat{C}_s^2\right) + \chi G^{1-\sigma}\left(\hat{G}_s + \frac{1}{2}(1-\sigma)\hat{G}_s^2\right) \right. \\ & \left. -N^{1+\varphi}\left(\hat{N}_s + \frac{1}{2}(1+\varphi)\hat{N}_s^2\right)\right) + t.i.p + O(3), \end{aligned} \tag{93}$$

where $t.i.p$ stands for terms that are independent of policy and hence irrelevant for ranking alternative policies. In this model, we have assumed that a subsidy makes the steady state efficient. Using the steady-state versions of Equations (70) to (71),

$$\frac{1}{C^\sigma} = \frac{\chi}{G^\sigma} = \frac{N^{1+\varphi}}{(1-\alpha)Y}. \tag{94}$$

This means that

$$\chi = \frac{G^\sigma}{C^\sigma}, \qquad N^{1+\varphi} = \frac{(1-\alpha)}{\theta_c}C^{1-\sigma}.$$

The social welfare function can be re-written as:

$$\begin{aligned} U_t = {} & \frac{C^{1-\sigma}}{\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\theta_c\hat{C}_s + \theta_g\hat{G}_s + \frac{1}{2}(1-\sigma)\left(\theta_c\hat{C}_s^2 + \theta_g\hat{G}_s^2\right) \right. \\ & \left. -(1-\alpha)\left(\hat{N}_s + \frac{1}{2}(1+\varphi)\hat{N}_s^2\right)\right) + t.i.p + O(3). \end{aligned} \tag{95}$$

Next, we remove the first order terms from the social welfare function. We use the second-order approximated goods market clearing condition and the production function to eliminate these terms. First the aggregate production functions can be approximated up to the second order as follows:

$$\begin{aligned} \hat{Y}_t + \frac{1}{2}\hat{Y}_t^2 = {} & -\hat{\Delta}_t + \alpha\left(\hat{K}_t + \frac{1}{2}\hat{K}_t^2\right) + (1-\alpha)\left(\hat{N}_t + \frac{1}{2}\hat{N}_t^2\right) - \frac{\alpha(1-\alpha)}{2}\hat{K}_t^2 \\ & -\frac{\alpha(1-\alpha)}{2}\hat{N}_t^2 + \hat{A}_t\left(\alpha\hat{K}_t + (1-\alpha)\hat{N}_t\right) + \alpha(1-\alpha)\hat{K}_t\hat{N}_t \\ & +t.i.p + O(3). \end{aligned} \tag{96}$$

It can be shown (see Woodford 2003a, chapter 6) that:

$$\Delta_t \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\epsilon_t} di = 1 + \frac{\epsilon}{2}Var_i(\hat{P}_t(i)) + O(3). \tag{97}$$

The goods market clearing condition can be approximated to the second order as follows:

$$\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2 = \theta_c\left(\hat{C}_t + \frac{1}{2}\hat{C}_t^2\right) + \theta_g\left(\hat{G}_t + \frac{1}{2}\hat{G}_t^2\right) + \theta_i\left(\hat{I}_t + \frac{1}{2}\hat{I}_t^2\right) + O(3), \tag{98}$$

where the investment terms can be written in terms of capital, using the second-order approximation of the capital accumulation equation:

$$\left(\hat{I}_t + \frac{1}{2}\hat{I}_t^2\right) = \frac{1}{\delta}\left(\hat{K}_{t+1} + \frac{1}{2}\hat{K}_{t+1}^2\right) - \frac{1-\delta}{\delta}\left(\hat{K}_t + \frac{1}{2}\hat{K}_t^2\right) - \frac{1}{2}J''\left(\Delta\hat{K}_{t+1}\right)^2 + O(3) \tag{99}$$

Equations (96), (98) and (99) are combined to obtain:

$$\begin{aligned}
& \theta_c\hat{C}_t + \theta_g\hat{G}_t - (1-\alpha)\hat{N}_t \\
= {} & \alpha\hat{K}_t + \frac{1}{2}\alpha^2\hat{K}_t^2 + \frac{(1-\alpha)^2}{2}\hat{N}_t^2 + \hat{A}_t\left(\alpha\hat{K}_t + (1-\alpha)\hat{N}_t\right) + \alpha(1-\alpha)\hat{K}_t\hat{N}_t \\
& -\frac{\epsilon}{2}Var_i(\hat{P}_t(i)) - \frac{\theta_c}{2}\hat{C}_t^2 - \frac{\theta_g}{2}\hat{G}_t^2 - \frac{\theta_i}{\delta}\left(\hat{K}_{t+1} + \frac{1}{2}\hat{K}_{t+1}^2\right) + \frac{\theta_i(1-\delta)}{\delta}\left(\hat{K}_t + \frac{1}{2}\hat{K}_t^2\right) \\
& +\frac{\theta_i}{2\delta}J''\left(\Delta\hat{K}_{t+1}\right)^2 + t.i.p + O(3).
\end{aligned} \tag{100}$$

This expression is substituted into (95) as follows:

$$\begin{aligned}
U_t = {} & \frac{C^{1-\sigma}}{\theta_c}E_t\sum_{s=t}^{\infty}\beta^{s-t}\begin{pmatrix}
\alpha\hat{K}_s + \frac{1}{2}\alpha^2\hat{K}_s^2 + \frac{(1-\alpha)^2}{2}\hat{N}_s^2 + \hat{A}_s\left(\alpha\hat{K}_s + (1-\alpha)\hat{N}_s\right) \\
+\alpha(1-\alpha)\hat{K}_s\hat{N}_s - \frac{\theta_c}{2}\hat{C}_s^2 - \frac{\theta_g}{2}\hat{G}_s^2 - \frac{\theta_i}{\delta}\left(\hat{K}_{s+1} + \frac{1}{2}\hat{K}_{s+1}^2\right) \\
+\frac{\theta_i(1-\delta)}{\delta}\left(\hat{K}_s + \frac{1}{2}\hat{K}_s^2\right) - \frac{1}{2}(1-\alpha)(1+\varphi)\hat{N}_s^2 - \frac{\epsilon}{2}Var_i(\hat{P}_s(i)) \\
+\frac{\theta_i}{2\delta}J''\left(\Delta\hat{K}_{s+1}\right)^2 + \frac{1}{2}(1-\sigma)\left(\theta_c\hat{C}_s^2 + \theta_g\hat{G}_s^2\right)
\end{pmatrix} \\
& +t.i.p + O(3),
\end{aligned} \tag{101}$$

$$\begin{aligned}
= {} & -\frac{C^{1-\sigma}}{2\theta_c}E_t\sum_{s=t}^{\infty}\beta^{s-t}\begin{pmatrix}
-2\left[\left(\alpha + \frac{\theta_i(1-\delta)}{\delta}\right)\hat{K}_s - \frac{\theta_i}{\delta}\hat{K}_{s+1}\right] - \left(\alpha^2 + \frac{\theta_i(1-\delta)}{\delta}\right)\hat{K}_s^2 \\
-2\hat{A}_s\left(\alpha\hat{K}_s + (1-\alpha)\hat{N}_s\right) - 2\alpha(1-\alpha)\hat{K}_s\hat{N}_s \\
+\sigma\theta_c\hat{C}_s^2 + \sigma\theta_g\hat{G}_s^2 + \frac{\theta_i}{\delta}\hat{K}_{s+1}^2 + (1-\alpha)(\alpha+\varphi)\hat{N}_s^2 \\
+\epsilon Var_i(\hat{P}_s(i)) - \frac{\theta_i}{\delta}J''\left(\Delta\hat{K}_{s+1}\right)^2
\end{pmatrix} \\
& +t.i.p + O(3).
\end{aligned} \tag{102}$$

Notice that

$$\left(\alpha + \frac{\theta_i(1-\delta)}{\delta}\right)\hat{K}_s - \frac{\theta_i}{\delta}\hat{K}_{s+1} = \frac{\theta_i}{\delta}\left[\left(\frac{\alpha\delta}{\theta_i} + 1 - \delta\right)\hat{K}_s - \hat{K}_{s+1}\right] = \frac{\theta_i}{\delta}\left[\frac{1}{\beta}\hat{K}_s - \hat{K}_{s+1}\right],$$

where the last equality follows from the steady-state version of Equation (72). Hence,

$$E_t \sum_{s=t}^{\infty} \beta^{s-t} \left[\left(\alpha + \frac{\theta_i(1-\delta)}{\delta} \right) \hat{K}_s - \frac{\theta_i}{\delta} \hat{K}_{s+1} \right] = E_t \sum_{s=t}^{\infty} \beta^{s-t} \left[\frac{1}{\beta} \hat{K}_s - \hat{K}_{s+1} \right] = \frac{1}{\beta} \hat{K}_t, \quad (103)$$

where the initial stock of capital K_t is independent of the policy.

Moreover, we can eliminate $\hat{A}_t$ using the production function and the intratemporal tradeoff between consumption and labour supply as follows:

$$\begin{aligned} \hat{A}_t &= \hat{Y}_t^n - \alpha \hat{K}_t^n - (1-\alpha)\hat{N}_t^n, \\ &= \left(\sigma \hat{C}_t^n + (1+\varphi)\hat{N}_t^n \right) - \alpha \hat{K}_t^n - (1-\alpha)\hat{N}_t^n, \\ &= (\alpha+\varphi)\hat{N}_t^n + \sigma \hat{C}_t^n - \alpha \hat{K}_t^n. \end{aligned} \quad (104)$$

Hence,

$$\begin{aligned} & -2\hat{A}_s \left(\alpha \hat{K}_s + (1-\alpha)\hat{N}_s \right) \\ =\ & -2 \left[(\alpha+\varphi)\hat{N}_s^n + \sigma \hat{C}_s^n - \alpha \hat{K}_s^n \right] \left(\alpha \hat{K}_s + (1-\alpha)\hat{N}_s \right), \\ =\ & -2\alpha(\alpha+\varphi)\hat{K}_s\hat{N}_s^n - 2(1-\alpha)(\alpha+\varphi)\hat{N}_s\hat{N}_s^n + 2\alpha^2\hat{K}_s\hat{K}_s^n + 2\alpha(1-\alpha)\hat{N}_s\hat{K}_s^n \\ & -2\left(\alpha\hat{K}_s + (1-\alpha)\hat{N}_s \right) \sigma\hat{C}_s^n, \\ =\ & -2\alpha(\alpha+\varphi)\hat{K}_s\hat{N}_s^n - 2(1-\alpha)(\alpha+\varphi)\hat{N}_s\hat{N}_s^n + 2\alpha^2\hat{K}_s\hat{K}_s^n + 2\alpha(1-\alpha)\hat{N}_s\hat{K}_s^n \\ & -2\left(\hat{Y}_s - \hat{A}_s \right) \sigma\hat{C}_s^n, \\ =\ & -2\alpha(\alpha+\varphi)\hat{K}_s\hat{N}_s^n - 2(1-\alpha)(\alpha+\varphi)\hat{N}_s\hat{N}_s^n + 2\alpha^2\hat{K}_s\hat{K}_s^n + 2\alpha(1-\alpha)\hat{N}_s\hat{K}_s^n \\ & -2\left(\theta_c\hat{C}_s + \frac{\theta_i}{\delta}(\hat{K}_{s+1} - (1-\delta)\hat{K}_s) + \theta_g\hat{G}_s \right) \sigma\hat{C}_s^n + t.i.p, \\ =\ & -2\alpha(\alpha+\varphi)\hat{K}_s\hat{N}_s^n - 2(1-\alpha)(\alpha+\varphi)\hat{N}_s\hat{N}_s^n + 2\alpha^2\hat{K}_s\hat{K}_s^n + 2\alpha(1-\alpha)\hat{N}_s\hat{K}_s^n \\ & -2\sigma\theta_c\hat{C}_s\hat{C}_s^n - 2\sigma\frac{\theta_i}{\delta}(\hat{K}_{s+1} - (1-\delta)\hat{K}_s)\hat{C}_s^n - 2\sigma\theta_g\hat{G}_s\hat{G}_s^n + t.i.p. \end{aligned} \quad (105)$$

Equation (105) is substituted into the social welfare function (102) as follows:

$$\begin{aligned} U_t \ =\ & -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\begin{array}{c} -\left(\alpha^2 + \frac{\theta_i(1-\delta)}{\delta} \right) \hat{K}_s^2 - 2\alpha(\alpha+\varphi)\hat{K}_s\hat{N}_s^n \\ -2(1-\alpha)(\alpha+\varphi)\hat{N}_s\hat{N}_s^n + 2\alpha^2\hat{K}_s\hat{K}_s^n + 2\alpha(1-\alpha)\hat{N}_s\hat{K}_s^n \\ -2\sigma\theta_c\hat{C}_s\hat{C}_s^n - 2\sigma\frac{\theta_i}{\delta}(\hat{K}_{s+1} - (1-\delta)\hat{K}_s)\hat{C}_s^n - 2\sigma\theta_g\hat{G}_s\hat{G}_s^n \\ -2\alpha(1-\alpha)\hat{K}_s\hat{N}_s + \sigma\theta_c\hat{C}_s^2 + \sigma\theta_g\hat{G}_s^2 + \frac{\theta_i}{\delta}\hat{K}_{s+1}^2 \\ +(1-\alpha)(\alpha+\varphi)\hat{N}_s^2 + \epsilon Var_i(\hat{P}_s(i)) - \frac{\theta_i}{\delta}J'' \left(\Delta\hat{K}_{t+1} \right)^2 \end{array} \right) \\ & +t.i.p + O(3), \end{aligned} \quad (106)$$

$$= -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\begin{array}{c} -\frac{\theta_i(1-\delta)}{\delta}\hat{K}_s^2 - \alpha^2 \left(\hat{K}_s^2 - 2\hat{K}_s\hat{K}_s^2\right) - 2\alpha(1+\varphi)\hat{K}_s\hat{N}_s^n \\ +(1-\alpha)(\alpha+\varphi)\left(\hat{N}_s^2 - 2\hat{N}_s\hat{N}_s^n\right) - 2\sigma\frac{\theta_i}{\delta}(\hat{K}_{s+1} - (1-\delta)\hat{K}_s)\hat{C}_s^n \\ -2\alpha(1-\alpha)\left(\hat{K}_s\hat{N}_s - \hat{K}_s\hat{N}_s^n - \hat{N}_s\hat{K}_s^n\right) + \sigma\theta_c\left(\hat{C}_s^2 - 2\hat{C}_s\hat{C}_s^n\right) \\ +\sigma\theta_g\left(\hat{G}_s^2 - 2\hat{G}_s\hat{G}_s^2\right) + \frac{\theta_i}{\delta}\hat{K}_{s+1}^2 + \epsilon Var_i(\hat{P}_s(i)) - \frac{\theta_i}{\delta}J''\left(\Delta\hat{K}_{t+1}\right)^2 \end{array} \right)$$
$$+t.i.p + O(3), \tag{107}$$

$$= -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\begin{array}{c} \sigma\theta_c c_s^2 + \sigma\theta_g g_s^2 + (1-\alpha)(\alpha+\varphi)n_s^2 - \alpha^2 k_s^2 - 2\alpha(1-\alpha)k_s n_s \\ +\epsilon Var_i(\hat{P}_s(i)) - \frac{\theta_i(1-\delta)}{\delta}\hat{K}_s^2 - 2\sigma\frac{\theta_i}{\delta}(\hat{K}_{s+1} - (1-\delta)\hat{K}_s)\hat{C}_s^n \\ -2\alpha(1+\varphi)\hat{K}_s\hat{N}_s^n + \frac{\theta_i}{\delta}\hat{K}_{s+1}^2 - \frac{\theta_i}{\delta}J''\left(\Delta\hat{K}_{t+1}\right)^2 \end{array} \right)$$
$$+t.i.p + O(3). \tag{108}$$

Note that:

$$\begin{aligned}
& E_t \sum_{s=t}^{\infty} \beta^{s-t} 2\alpha(1+\varphi)\hat{K}_s\hat{N}_s^n \\
= \; & E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(2\alpha\hat{K}_s\hat{Y}_s^n - 2\sigma\alpha\hat{K}_s\hat{C}_s^n\right), \\
= \; & E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(2\alpha\beta\hat{K}_{s+1}\hat{Y}_{s+1}^n - 2\sigma\alpha\hat{K}_s\hat{C}_s^n\right) + 2\alpha\hat{K}_t\hat{Y}_t^n, \\
= \; & E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(2\alpha\beta\hat{K}_{s+1}\hat{Y}_{s+1}^n - 2\alpha\beta\hat{K}_{s+1}\hat{K}_{s+1}^n + 2\alpha\beta\hat{K}_{s+1}\hat{K}_{s+1}^n - 2\sigma\alpha\hat{K}_s\hat{C}_s^n\right) + t.i.p, \\
= \; & E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(2\alpha\beta\hat{K}_{s+1}\left(\hat{Y}_{s+1}^n - \hat{K}_{s+1}^n\right) + 2\alpha\hat{K}_s\hat{K}_s^n - 2\sigma\alpha\hat{K}_s\hat{C}_s^n\right) - 2\alpha\hat{K}_t\hat{K}_t^n + t.i.p, \\
= \; & E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\begin{array}{c} \frac{2\alpha\beta}{1-(1-\delta)\beta}\hat{K}_{s+1}\left(\sigma\hat{C}_{s+1}^n - \sigma\hat{C}_s^n - J''\Delta\hat{K}_{t+1}^n + \beta J''\Delta\hat{K}_{t+2}^n\right) \\ +2\alpha\hat{K}_s\hat{K}_s^n - 2\sigma\alpha\hat{K}_s\hat{C}_s^n \end{array} \right) \\
& +t.i.p,
\end{aligned} \tag{109}$$

where the first equality makes use of the log-linearsed labour supply condition in the flexible price model, $(1+\varphi)\hat{N}_t^n = \hat{Y}_t^n - \sigma\hat{C}_t^n$, and the final equality makes use of the log-linearised investment demand condition in the flexible price model:

$$\sigma\hat{C}_t^n = E_t\left[\sigma\hat{C}_{t+1}^n - J''\Delta\hat{K}_{t+1}^n + \beta J''\Delta\hat{K}_{t+2}^n - (1-(1-\delta)\beta)\left(\hat{Y}_{t+1}^n - \hat{K}_{t+1}^n\right)\right].$$

Moreover, the investment demand condition in the steady state results in $\frac{\alpha\beta}{1-(1-\delta)\beta} = \theta_i/\delta$.

We substitute Equation (109) into the social welfare function (108) as follows:

$$
\begin{aligned}
U_t &= -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \begin{pmatrix} \theta_c c_s^2 + \theta_g g_s^2 + (1-\alpha)(\alpha+\varphi) n_s^2 - \alpha^2 k_s^2 - 2\alpha(1-\alpha) k_s n_s \\ +\epsilon Var_i(\hat{P}_s(i)) - \frac{\theta_i(1-\delta)}{\delta} \hat{K}_s^2 - 2\sigma \frac{\theta_i}{\delta} (\hat{K}_{s+1} - (1-\delta)\hat{K}_s) \hat{C}_s^n \\ -\frac{2\theta_i}{\delta} \hat{K}_{s+1} \left(\sigma \hat{C}_{s+1}^n - \sigma \hat{C}_s^n - J'' \Delta \hat{K}_{s+1}^n + \beta J'' \Delta \hat{K}_{s+2}^n \right) \\ -2\alpha \hat{K}_s \hat{K}_s^n + 2\sigma\alpha \hat{K}_s \hat{C}_s^n + \frac{\theta_i}{\delta} \hat{K}_{s+1}^2 - \frac{\theta_i}{\delta} J'' \left(\Delta \hat{K}_{s+1} \right)^2 \end{pmatrix} \\
&\quad + t.i.p + O(3), \qquad (110)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \begin{pmatrix} \theta_c c_s^2 + \theta_g g_s^2 + (1-\alpha)(\alpha+\varphi) n_s^2 - \alpha^2 k_s^2 - 2\alpha(1-\alpha) k_s n_s \\ +\epsilon Var_i(\hat{P}_s(i)) + 2\sigma \frac{\theta_i}{\delta} \left[\left((1-\delta) + \frac{\alpha\delta}{\theta_i} \right) \hat{K}_s \hat{C}_s^n - \hat{K}_{s+1} \hat{C}_{s+1}^n \right] \\ -\frac{\theta_i}{\delta} \left[\left((1-\delta) + \frac{\alpha\delta}{\theta_i} \right) \hat{K}_s^2 - \hat{K}_{s+1}^2 \right] + \alpha \hat{K}_s^2 - 2\alpha \hat{K}_s \hat{K}_s^n \\ -\frac{\theta_i}{\delta} J'' \left(2\hat{K}_{s+1} \left(\beta \Delta \hat{K}_{s+2}^n - \Delta \hat{K}_{s+1}^n \right) + \left(\Delta \hat{K}_{s+1} \right)^2 \right) \end{pmatrix} \\
&\quad + t.i.p + O(3). \qquad (111)
\end{aligned}
$$

Notice that:

$$
\begin{aligned}
& E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(2\hat{K}_{s+1} \left(\beta \Delta \hat{K}_{s+2}^n - \Delta \hat{K}_{s+1}^n \right) + \left(\Delta \hat{K}_{s+1} \right)^2 \right), \\
&= E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(2\hat{K}_s \left(\Delta \hat{K}_{s+1}^n \right) - 2\hat{K}_{s+1} \left(\Delta \hat{K}_{s+1}^n \right) + \left(\Delta \hat{K}_{s+1} \right)^2 \right) - 2\hat{K}_t \left(\Delta \hat{K}_{t+1}^n \right), \\
&= E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(2\hat{K}_s \left(\Delta \hat{K}_{s+1}^n \right) - 2\hat{K}_{s+1} \left(\Delta \hat{K}_{s+1}^n \right) + \hat{K}_{s+1}^2 - 2\hat{K}_s \hat{K}_{s+1} + \hat{K}_s^2 \right) + t.i.p, \\
&= E_t \sum_{s=t}^{\infty} \beta^{s-t} \begin{pmatrix} 2\hat{K}_s \left(\Delta \hat{K}_{s+1}^n \right) - 2\hat{K}_{s+1} \left(\Delta \hat{K}_{s+1}^n \right) \\ + \left(\hat{K}_{s+1} - \hat{K}_{s+1}^n \right)^2 + 2\hat{K}_{s+1} \hat{K}_{s+1}^n \\ -2 \left(\hat{K}_s - \hat{K}_s^n \right) \left(\hat{K}_{s+1} - \hat{K}_{s+1}^n \right) - 2\hat{K}_{s+1} \hat{K}_s^n - 2\hat{K}_s \hat{K}_{s+1}^n \\ + \left(\hat{K}_s - \hat{K}_s^n \right)^2 + 2\hat{K}_s \hat{K}_s^n \end{pmatrix} + t.i.p, \\
&= E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\Delta k_{s+1} \right)^2 + t.i.p. \qquad (112)
\end{aligned}
$$

The utility-based welfare function is just:

$$
\begin{aligned}
&= -\frac{C^{1-\sigma}}{2\theta_c} E_t \sum_{s=t}^{\infty} \beta^{s-t} \begin{pmatrix} \sigma\theta_c c_s^2 + \sigma\theta_g g_s^2 + \varphi(1-\alpha) n_s^2 \\ +\alpha(1-\alpha)(n_s - k_s)^2 - \frac{\theta_i}{\delta} J'' \left(\Delta k_{s+1} \right)^2 + \epsilon Var_i(\hat{P}_s(i)) \end{pmatrix} \\
&\quad + t.i.p + O(3). \qquad (113)
\end{aligned}
$$

One can use the production function in gap form ($y_t = \alpha k_t + (1-\alpha) n_t$) to write the

the welfare function in terms of output as follows:

$$U_t = -\frac{C^{1-\sigma}}{2\theta_c}E_t\sum_{s=t}^{\infty}\beta^{s-t}\left(\begin{array}{c}\sigma\theta_c c_s^2+\sigma\theta_g g_s^2+\frac{\alpha+\varphi}{(1-\alpha)}y_s^2-2\frac{\alpha(1+\varphi)}{(1-\alpha)}y_s k_s\\ +\frac{\alpha(1+\alpha\varphi)}{(1-\alpha)}k_s^2-\frac{\theta_i}{\delta}J''\left(\Delta k_{s+1}\right)^2+\epsilon Var_i(\hat{P}_s(i))\end{array}\right) +t.i.p+O(3). \quad (114)$$

Lastly, regarding the price dispersion, Steinsson (2003) shows that:

$$\sum_{t=0}^{\infty}\beta^t Var_i(\hat{P}_t(i)) = \frac{1}{1-\gamma\beta}\sum_{t=0}^{\infty}\beta^t\left(\begin{array}{c}\frac{\gamma}{1-\gamma}\pi_t^2-\frac{2\omega\eta}{1-\omega}\Delta\pi_t y_{t-1}\\ +\frac{\omega}{(1-\gamma)(1-\omega)}\Delta\pi_t^2+\frac{(1-\gamma)\omega\eta^2}{1-\omega}y_{t-1}^2\end{array}\right) +t.i.p+O(3). \quad (115)$$

Hence the social loss function L is:

$$L_t = -\frac{C^{1-\sigma}}{2\theta_c}E_t\sum_{s=t}^{\infty}\beta^{s-t}\left(\begin{array}{c}\sigma\theta_c c_s^2+\sigma\theta_g g_s^2+\frac{\alpha+\varphi}{(1-\alpha)}y_s^2\\ -2\frac{\alpha(1+\varphi)}{(1-\alpha)}y_s k_s+\frac{\alpha(1+\alpha\varphi)}{(1-\alpha)}k_s^2-\frac{\theta_i}{\delta}J''\left(\Delta k_{s+1}\right)^2\\ +\frac{\epsilon}{\kappa}\pi_s^2+2\frac{\epsilon}{\kappa}\phi_1\left(\Delta\pi_s\right)y_{s-1}+\frac{\epsilon}{\kappa}\phi_2\left(\Delta\pi_s\right)^2+\frac{\epsilon}{\kappa}\phi_3 y_{s-1}^2\end{array}\right) +t.i.p+O(3), \quad (116)$$

where

$$\kappa=\frac{(1-\gamma\beta)(1-\gamma)}{\gamma}, \qquad \phi_1=-\frac{\omega\eta(1-\gamma)}{\gamma(1-\omega)}, \qquad \phi_2=\frac{\omega}{\gamma(1-\omega)}, \qquad \phi_3=\frac{(1-\gamma)^2\omega\eta^2}{\gamma(1-\omega)}.$$

To derive the social welfare function in the model without capital, set $\alpha=0$, $\theta_i=0$, $\theta_c=\theta$ and $\theta_g=(1-\theta)$ to obtain:

$$L_t = -\frac{C^{1-\sigma}}{2\theta_c}E_t\sum_{s=t}^{\infty}\beta^{s-t}\left(\begin{array}{c}\sigma\theta c_s^2+\sigma(1-\theta)g_s^2+\varphi y_s^2+\frac{\epsilon}{\kappa}\pi_s^2\\ +2\frac{\epsilon}{\kappa}\phi_1\left(\Delta\pi_s\right)y_{s-1}+\frac{\epsilon}{\kappa}\phi_2\left(\Delta\pi_s\right)^2+\frac{\epsilon}{\kappa}\phi_3 y_{s-1}^2\end{array}\right) +t.i.p+O(3). \quad (117)$$

When there is no persistence in inflation, further set $\omega=\phi_1=\phi_2=\phi_3=0$ to obtain:

$$L_t = -\frac{C^{1-\sigma}}{2\theta_c}E_t\sum_{s=t}^{\infty}\beta^{s-t}\left(\sigma\theta c_s^2+\sigma(1-\theta)g_s^2+\varphi y_s^2+\frac{\epsilon}{\kappa}\pi_s^2\right) +t.i.p+O(3). \quad (118)$$

A.4 Solution Methods

The policymaker chooses the policy to minimise the quadratic loss function subject to the linear constraints of the economy. The loss function is

$$L_t=E_t\sum_{s=0}^{\infty}\beta^s\left[\begin{array}{cc}x_{t+s}' & u_{t+s}'\end{array}\right]\left[\begin{array}{cc}Q & U\\ U' & R\end{array}\right]\left[\begin{array}{c}x_{t+s}\\ u_{t+s}\end{array}\right],$$

where $x_t = [x'_{1t}\ x'_{2t}]'$. x_{1t} is a $n_1 \times 1$ vector of predetermined variables, x_{2t} an $n_2 \times 1$ vector of forward-looking variables, u_t a $n_u \times 1$ vector of control variables. Q and R are symmetric matrices. Q can be further partitioned into

$$Q = \begin{bmatrix} Q_{1,1} & Q_{1,2} \\ Q'_{1,2} & Q_{2,2} \end{bmatrix}$$

where $Q_{1,1}$ is $n1 \times n_1$, $Q_{1,2}$ is $n_1 \times n_2$ and $Q_{2,2}$ is $n_2 \times n_2$.

The constraints are

$$\begin{bmatrix} I & \mathbf{0}_{n_1 \times n_2} \\ \mathbf{0}_{n_2 \times n_1} & H \end{bmatrix} \begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_t + \begin{bmatrix} \xi_{t+1} \\ \mathbf{0}_{n_2 \times 1} \end{bmatrix},$$

where x_{10} is given.

Under commitment, the policymaker chooses an optimal path from time t onwards and the public believes that the policymaker will stick to this path. Since the policymaker can affect the public's expectation of future variables, the optimal commitment policy is the solution of the standard Lagrangian problem. For this problem, we find it convenient to write the constraints in the following form:

$$\bar{H} \begin{bmatrix} x_{1t+1} \\ E_t x_{2t+1} \\ E_t u_{t+1} \end{bmatrix} = \bar{A} \begin{bmatrix} x_{1t} \\ x_{2t} \\ u_t \end{bmatrix} + \begin{bmatrix} \xi_{t+1} \\ \mathbf{0}_{n_2 \times 1} \\ \mathbf{0}_{n_u \times 1} \end{bmatrix}, \tag{119}$$

where we define

$$\bar{H} \equiv \begin{bmatrix} I & \mathbf{0}_{n_1 \times n_2} & \mathbf{0}_{n_1 \times n_u} \\ \mathbf{0}_{n_2 \times n_1} & H & \mathbf{0}_{n_2 \times n_u} \end{bmatrix}, \qquad \bar{A} \equiv \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{bmatrix}, \qquad W \equiv \begin{bmatrix} Q & U \\ U' & R \end{bmatrix}.$$

The Lagrangian is

$$\max_{[x'_{1s}, x'_{2s}, u'_s]'} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\begin{bmatrix} x_{1s} \\ x_{2s} \\ u_s \end{bmatrix}' W \begin{bmatrix} x_{1s} \\ x_{2s} \\ u_s \end{bmatrix} + \begin{bmatrix} \lambda^{x_1}_{s+1} \\ \lambda^{x_2}_{s+1} \end{bmatrix}' \left(\bar{H} \begin{bmatrix} x_{1s+1} \\ x_{2s+1} \\ u_{s+1} \end{bmatrix} - \bar{A} \begin{bmatrix} x_{1s} \\ x_{2s} \\ u_s \end{bmatrix} \right) \right),$$

where $\lambda^{x_1}_{s+1}$ and $\lambda^{x_2}_{s+1}$ are the Lagrange multipliers for the predetermined variables and the forward-looking variables respectively.

The first order conditions are:

$$2W \begin{bmatrix} x_{1t} \\ x_{2t} \\ u_t \end{bmatrix} + \frac{1}{\beta} \bar{H}' \begin{bmatrix} \lambda^{x_1}_t \\ \lambda^{x_2}_t \end{bmatrix} - \bar{A}' \begin{bmatrix} \lambda^{x_1}_{t+1} \\ \lambda^{x_2}_{t+1} \end{bmatrix} = \mathbf{0}_{(n_1+n_2+n_u) \times 1}. \tag{120}$$

The Lagrange multipliers for the predetermined variables are forward-looking variables and the Lagrange multipliers for the forward-looking variables are predetermined. We set the Lagrange multipliers for the forward-looking variables in initial period t to be zero ($\lambda_0^{x_2} = 0$). This ensures that the initial policy is optimal and that the policymaker has no incentive to renege from this policy at time 0.

Then, the constraints (119), the first order conditions (120), together with the initial conditions for the predetermined variables fully describe the optimal commitment solution. The system can be written succinctly as follows:

$$\begin{bmatrix} \bar{H} & \mathbf{0} \\ \mathbf{0} & \bar{A}' \end{bmatrix} \begin{bmatrix} x_{1t+1} \\ x_{2t+1} \\ u_{t+1} \\ \lambda_{t+1}^{x_1} \\ \lambda_{t+1}^{x_2} \end{bmatrix} = \begin{bmatrix} \bar{A} & \mathbf{0} \\ 2W & \frac{1}{\beta}\bar{H}' \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \\ u_t \\ \lambda_t^{x_1} \\ \lambda_t^{x_2} \end{bmatrix} + \begin{bmatrix} \xi_{t+1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

where the predetermined variables are $\{x_{1t},\ \lambda_t^{x_2}\}$ and the forward-looking variables are $\{x_{2t},\ u_t,\ \lambda_t^{x_1}\}$. The solution is found by generalised Schur decomposition.

We write the models in this paper in the form above. In the model without capital and inflation persistence, the vectors of variables $\{x_{1t}, x_{2t}, u_t, \xi_t\}$, the matrices $\{A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, Q_{11}, Q_{12}, Q_{22}, U_1, U_2, R\}$ and sizes $\{n_1, n_2, n, k\}$ are:

$$\begin{aligned}
x_{1t} &= \left[\hat{\mu}_t\ \hat{B}_t\right]',\ \ x_{2t} = [\pi_t\ y_t\ c_t]',\ \ u_t = [r_t\ g_t]',\ \ \xi_t = [\epsilon_{\mu,t}\ 0]', \\
H &= \begin{bmatrix} -\beta & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -\sigma \end{bmatrix}, A_{11} = \begin{bmatrix} \rho_\mu & 0 \\ 0 & \frac{1}{\beta} \end{bmatrix},\ \ A_{12} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{\beta} & -\frac{\tau}{\beta B} & 0 \end{bmatrix},\ \ A_{21} = \begin{bmatrix} \kappa & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
A_{22} &= \begin{bmatrix} -1 & \kappa\varphi & \kappa\sigma \\ 0 & -1 & \theta \\ 0 & 0 & -\sigma \end{bmatrix},\ \ B_1 = \begin{bmatrix} 0 & 0 \\ 1 & \frac{(1-\theta)}{\beta B} \end{bmatrix},\ \ B_2 = \begin{bmatrix} 0 & 0 \\ 1 & (1-\theta) \\ -1 & 0 \end{bmatrix}, \\
Q_{11} &= \mathbf{0}_{2\times 2},\ \ Q_{12} = \mathbf{0}_{2\times 3},\ \ Q_{22} = \begin{bmatrix} \frac{\epsilon}{\kappa} & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & \sigma\theta \end{bmatrix}, \\
U_1 &= \mathbf{0}_{2\times 2},\ \ U_2 = \mathbf{0}_{2\times 3},\ \ R = \begin{bmatrix} 0 & 0 \\ 0 & (1-\theta)\sigma \end{bmatrix}, \\
n_1 &= 2,\ \ n_2 = 3,\ \ n_u = 2.
\end{aligned}$$

We write our model in the form above. The vectors of variables $\{x_{1t}, x_{2t}, u_t, \xi_t\}$, the matrices

$\{A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, Q_{11}, Q_{12}, Q_{22}, U_1, U_2, R\}$ and sizes $\{n_1, n_2, n, k\}$ are:

$$x_{1t} = \left[\hat{\mu}_t \ \pi_{t-1} \ y_{t-1} \ \hat{B}_t \ k_t\right]', \qquad x_{2t} = [\pi_t \ y_t \ c_t \ k_{t+1}]', \ \ u_t = [r_t \ g_t]', \ \ \xi_t = [\epsilon_{\mu,t} \ 0 \ 0 \ 0]',$$

$$H = \begin{bmatrix} -\chi^{FL}\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -\sigma & 0 \\ -1 & -(1-\beta(1-\delta))\frac{1+\varphi}{1-\alpha} & -\sigma(1-\beta(1-\delta)) & \beta J'' \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} \rho_\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\beta} & -\frac{\tau}{\beta B} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} \kappa_\mu & \chi^{BL} & \kappa_{y1} & 0 & -\kappa_k \\ 0 & 0 & 0 & 0 & -\frac{\theta_i(1-\delta)}{\delta} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -J'' \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} -1 & \kappa_{y0} & \kappa_c & 0 \\ 0 & -1 & \theta_c & \frac{\theta_i}{\delta} \\ 0 & 0 & -\sigma & 0 \\ 0 & 0 & 0 & -\left((1-\beta(1-\delta))\left(\frac{1+\alpha\varphi}{1-\alpha}\right) - (1+\beta)J''\right) \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & \frac{\theta_g}{\beta B} \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & \theta_g \\ -1 & 0 \\ -1 & 0 \end{bmatrix},$$

$$Q_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon}{\kappa}\phi_2 & -\frac{\epsilon}{\kappa}\phi_1 & 0 & 0 \\ 0 & -\frac{\epsilon}{\kappa}\phi_1 & \frac{\epsilon}{\kappa}\phi_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha(1+\alpha\varphi)}{1-\alpha} - J''\frac{\theta_i}{\delta} \end{bmatrix}, \quad Q_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{\epsilon}{\kappa}\phi_2 & 0 & 0 & 0 \\ \frac{\epsilon}{\kappa}\phi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\alpha(1+\varphi)}{1-\alpha} & 0 & J''\frac{\theta_i}{\delta} \end{bmatrix},$$

$$Q_{22} = \begin{bmatrix} \frac{\epsilon}{\kappa}(1+\phi_2) & 0 & 0 & 0 \\ 0 & \frac{\alpha+\varphi}{1-\alpha} & 0 & 0 \\ 0 & 0 & \sigma\theta_c & 0 \\ 0 & 0 & 0 & -J''\frac{\theta_i}{\delta} \end{bmatrix}, \quad U_1 = \mathbf{0}_{5\times 2}, \quad U_2 = \mathbf{0}_{4\times 2},$$

$$R = \begin{bmatrix} 0 & 0 \\ 0 & \sigma\theta_g \end{bmatrix}, \quad n_1 = 5, \quad n_2 = 4, \quad n_u = 2.$$