Local Intermediation, Dynamic Information Aggregation, and Excessive Trading*

(Incomplete: Comments are welcome)

Kei Kawakami†

November 14th, 2013

Abstract

This paper presents a dynamic equilibrium model of information aggregation which occurs through locally intermediated trading in a large population. We analyze how dispersed information is aggregated over time, and how it affects allocations overtime. The model exhibits excess trading due to information aggregation. We show that in the absence of private signals, dynamic trading leads to the ex post efficient allocation as the number of trading rounds increases. With private signals, information gets aggregated over time, but the allocation may diverge away from the efficient allocation. In some cases, information aggregation can create excessive trading, increasing the dispersion of the allocation.

Keywords: Asymmetric information, Dynamic trading, Information aggregation, Post-trade efficiency.

*I thank Pierre-Olivier Weill, Ichiro Obara, Moritz Meyer-ter-Vehn and seminar participants at Monash and the 2nd UCLA alumni conference for helpful comments. All errors are mine. This project is financially supported by the Faculty of Business and Economics, University of Melbourne.

†Department of Economics, University of Melbourne, e-mail: keik@unimelb.edu.au
1 Introduction

We present a dynamic equilibrium model of information sharing which occurs through locally intermediated trading in a large population. We analyze how dispersed information is aggregated over time, and how it affects allocations over time. The model is simple but captures a key feature of decentralized trading where traders interact only locally and due to limited public information dissemination they form dispersed beliefs about the entire economy.

The model economy has two assets and a continuum of risk neutral traders. A risky asset ("tree") has uncertain payoff. Traders are endowed with the different amount of the tree, and this endowment is private information. There is a convex cost of holding inventory of trees. This creates potential gains from reallocating (smoothing) the tree positions by using a non-risky asset ("money") as a means of exchange. Before a trading process starts, each trader receives an independent noisy signal about the payoff of the tree. Thus, each trader has two pieces of private information: the value and position of the tree. The trading is locally intermediated in the following sense. At each time, \( n + 1 \) traders are randomly matched where \( n \geq 1 \) is a finite fixed number. Each trader submits his demand for the tree to a market maker, which is explicitly conditioned on the unit price of the tree (the amount of money exchanged for one tree). Given \( n + 1 \) submitted orders, the market maker determines one price for each trader subject to a feasibility constraint. Thus, a market is "locally cleared" for each group of \( n + 1 \) traders. In each period, with some probability the game ends. As the trading process continues, traders learn information from transaction prices. The distribution of the traders' tree positions and their beliefs about the value of the tree in the entire economy both endogenously change over time. Each trader rationally forms an expectation about how these distributions evolve over time, and knows that he or she trades with a random sample of \( n \) traders whose types are drawn from these distributions. We show that in the absence of private signals about the tree, dynamic trading leads to the ex post efficient allocation as the number of trading rounds increases. With private signals, information is aggregated over time, but the allocation may diverge away from the
efficient allocation. In some cases, information aggregation can create excessive trading, increasing the dispersion of the allocation relative to its initial level. Figure 1 shows how the cross-sectional variance of the tree allocations changes over time for a particular parameter configuration. There is a period in which trading makes the tree allocation more dispersed.

![Figure 1: The variance of the tree allocation over time.](image)

Notes. The number of traders in each local trading is \( n + 1 = 4 \). The number of maximum trading rounds is \( T = 10 \). Larger gamma means that traders put more probability weights on future trading rounds.

**Related literature.** This paper contributes to the literature on information aggregation with equilibrium dynamics. Duffie, Malamud and Manso (2009, 2013) study the information dynamics in a search model. The information structure in this paper is similar to theirs, but in their model agents perfectly share their information when they are matched with the other agent. In our model, the information is aggregated through trading of a perfectly divisible asset and the information asymmetry between traders remains after trading. Amador and Weill (2012) study a learning dynamics associated with local interactions similar to our model, but there is no trading in their model. Our model has explicit asset trading, making it possible to study the interaction between allocational efficiency and learning efficiency. Ostrovsky (2012) and Iyer, Johari, and Moallemi (2011) study dynamic information aggregation among finite number of traders. While they focus on *public learning* where trading outcome is observed by everyone, we study *local learning* where a trading
outcome is known only to the party directly involved in the trading. Thus, our model is better suited for the analysis of decentralized markets characterized by local interactions and limited public information dissemination.

The most closely related work is Golosov, Lorenzoni, and Tsyvinski (2013). They study dynamic asset trading with asymmetric information where traders have two motives for trading, which makes information revelation non-trivial as in this paper. While they focus on bilateral bargaining, we use a trading rule that allows for multi-lateral trading including a bilateral case. Also, in their model, there are informed agents and uninformed agents, and learning is one-sided. In this paper, information is dispersed, so all traders learn in equilibrium. Finally, in their model, a static centralized mechanism can achieve the ex post efficient allocation. In contrast, the ex post efficient allocation is not implementable by any static mechanisms in our environment.\footnote{This is consistent with generic impossibility of ex post implementation (Jehiel, Meyer-Ter-Vehn, Moldovanu and Zame (2006)) with multi-dimensional private information.} This motivates us to investigate whether dynamic trading can achieve the ex post efficient allocation by aggregating information.

This paper is organized as follows. Section 2 describes an economic environment. Section 3 characterizes efficient allocations. We define the post-trade efficiency and compare it with the ex post efficiency. Section 4 introduces the trading rule. Section 5 studies dynamic trading. Section 6 concludes.

## 2 Model Environment

There is a measure one of continuum of traders indexed by $i \in I$. All traders have the same preference and trade a risky asset ("tree") with the uncertain unit payoff $v$ in exchange for a non-risky asset ("money"), which is a numeraire. The payoff $v$ is realized at time $t = T + 1$ and not known to anyone until then. Traders have a common prior that

$$v = \sqrt{uw_A} + \sqrt{1 - uv_B}$$
and \( v_A \) and \( v_B \) are independently drawn from a normal distribution with mean zero and variance \( \tau_v^{-1} \). Each trader has two types of private information: (i) endowment of the tree \( x_{i0} \), and (ii) a private signal \( s_{i0} \) about \( v_A \). A parameter \( u \in [0, 1] \) controls the degree of information asymmetry without affecting the ex ante variance of \( v \), where \( u = 0 \) captures a symmetric information case. The sum of endowments is the total amount of the tree in the economy. Each trader’s endowment is a realization of an independent normal random variable with mean zero and variance \( \tau_x^{-1} \). The private signal takes the form \( s_{i0} = v_A + \varepsilon_i \), where \( \varepsilon_i \) is unobserved noise in the signal, and follows a normal distribution with mean zero and variance \( \tau_{\varepsilon}^{-1} \). This means that \( \text{Corr} [s_{i0}, v] = \sqrt{\frac{\tau_v}{\tau_v + \tau_x}} \). To summarize, random variables \( v_A, v_B, \{x_{i0}, \varepsilon_i\}_{i \in I} \) are assumed to be normally and independently distributed with zero means, and variances

\[
\text{Var} (v_A) = \text{Var} (v_B) = \tau_v^{-1}, \quad \text{Var} (x_{i0}) = \tau_x^{-1}, \quad \text{Var} (\varepsilon_i) = \tau_{\varepsilon}^{-1}.
\]

Let \( b_{i0} \) be trader \( i \)'s initial money position. We assume that the net return on money is zero. Given an initial position \( (x_{i0}, b_{i0}) \) of the tree and the money, the payoff from adding \( q_i \) units of the tree and \( r_i \) units of the money is

\[
\pi_i (q_i, r_i; x_{i0}, b_{i0}) = v (q_i + x_{i0}) + r_i + b_{i0}.
\]

We call \( (q_i, r_i) \) a trade for trader \( i \). We assume that traders are risk neutral, but must incur a non-negative cost for holding a non-zero tree position:

\[
C_i (q_i + x_{i0}) = \frac{\rho}{2} (q_i + x_{i0})^2,
\]

\[\text{Signals that are negatively correlated with } v_A \text{ can be incorporated by letting } v = \text{sign}(u) \sqrt{|u|} v_A + \text{sign}(1-u) \sqrt{|1-u|} v_B \text{ with } u \in \mathbb{R}. \text{ Then } \text{Corr} [s_{i0}, v] = \frac{\text{sign}(u)}{|u| + |1-u|} \sqrt{\frac{\tau_v}{\tau_v + \tau_x}} \text{ and } u < 0 \text{ allows the negatively correlated signals. All the results presented below go through by replacing } \sqrt{u} \text{ with } \text{sign}(u) \sqrt{|u|} \text{ and } u \text{ with } |u|.\]

5
where $\rho \geq 0$ measures the adjustment cost of the tree position. Thus, the expected utility from a trade $(q_i, r_i)$ is

$$u_i(\pi_i) = E_i[\pi_i] - \frac{\rho}{2} (q_i + x_{i0})^2,$$  \hspace{1cm} (1)

where $E_i[\cdot]$ denotes trader $i$’s conditional expectation.\footnote{Our model environment is similar to Vives (2011) and Rostek and Weretka (2012), but it is simplified for tractability in a dynamic setup.} From (1), utility is transferable via money. Hence, we focus on the efficiency of the tree allocation. In this environment, trading can create social benefits through the second term in (1) by decreasing the cross-sectional dispersion of $x_{i0}$. On the other hand, the dispersion in beliefs $E_i[\pi_i]$ does not directly affect social welfare, unless it is systematically related to a mean belief $\int E_i[\pi_i]di$ or the dispersion of $q_i$. In particular, we are interested in the dispersion of the tree allocations

$$\int (q_i + x_{i0})^2 di.$$

(2)

In this paper, we take a size of local interactions as a primitive of the economy: trading must be among finite $n + 1$ traders who are randomly drawn from the population. The case $n = 1$ covers a case where trading is bilateral. The parameter $n$ measures the degree of “localness” of the trading process.

## 3 Efficient allocations

In this section, we study efficient allocations for a given trading group of the size $n$ to illustrate a basic trade-off between the information aggregation and the welfare in the model. We study two efficiency criteria. First, we consider the *ex post efficiency*. Second, we introduce a new efficiency criterion which we call the *post-trade efficiency*. Given the ex ante symmetry among traders, we focus on efficiency associated with equal Pareto weights. Also, we ignore the money allocation until it is used for implementation in section 4.
3.1 Ex post efficiency

Let \( q = \{q_i\}_{i=1}^{n+1} \) be the trade of the tree among \( n + 1 \) traders, and let \( s_0 = \{s_{i0}\}_{i=1}^{n+1} \) and \( x_0 = \{x_{i0}\}_{i=1}^{n+1} \). The ex post efficient allocation among \( n + 1 \) traders is \( q^{xp} + x_{i0} \), where the trade \( q^{xp} = \{q^{xp}_i\}_{i=1}^{n+1} \) is defined by

\[
q^{xp} = \arg \max_q \sum_{i=1}^{n+1} \left( E[\pi_i | s_0, x_0] - \frac{\rho}{2} (q_i + x_{i0})^2 \right)
\]

subject to \( \sum_{i=1}^{n+1} q_i = 0. \)

Note that beliefs in (3) are symmetric. The object of maximization is

\[
\sum_{i=1}^{n+1} \left\{ \sqrt{\mu} E[v_{A_i} | s_0, x_0] (q_i + x_{i0}) - \frac{\rho}{2} (q_i + x_{i0})^2 \right\} - \lambda \sum_{i=1}^{n+1} q_i,
\]

where \( E[v_{A_i} | s_0, x_0] = \frac{\tau_{x}}{\tau^{xp}} \sum_{i=1}^{n+1} s_{i0}, \tau^{xp} \equiv (\text{Var}[v_{A_i} | s_0, x_0])^{-1} = \tau_v + (n + 1) \tau_{x}, \)

and \( \lambda \) is a Lagrange multiplier. From the first order condition, it is immediate to check

\[
q^{xp}_i = \frac{1}{\rho} \left( \sqrt{\mu} E[v_{I_i} | s_0, x_0] - \lambda \right) - x_{i0},
\]

\[
\lambda = \sqrt{\mu} E[v_{I_i} | s_0, x_0] - \frac{1}{n+1} \sum_{i=1}^{n+1} x_{i0}
\]

\[
= \frac{\sqrt{\mu} \tau_{x} (n + 1)}{\tau^{xp}} s_0^{(n+1)} - \rho \hat{x}_0^{(n+1)},
\]

where \( s_0^{(n+1)} = \frac{1}{n+1} \sum_{i=1}^{n+1} s_{i0} \) and \( \tau_0^{(n+1)} = \frac{1}{n+1} \sum_{i=1}^{n+1} x_{i0} \) denote the average across \( n + 1 \) traders.

The next lemma characterizes this allocation.

Lemma 1
(a) The ex post efficient allocation among \(n + 1\) traders is

\[ q_{\pi}^x + x_0 = x_0^{(n+1)}, \]  

and the associated utility is

\[ \frac{\sqrt{u(n+1)}}{\tau_x} + n + 1 \times x_0^{(n+1)} - \frac{\rho}{2} \left( x_0^{(n+1)} \right)^2. \]  

(b) \( E[|q_{\pi}^x|] = \sqrt{\frac{2}{\pi} \cdot \frac{n}{n+1} \cdot \frac{1}{\tau_x}} \) and \( E \left[ \left( x_0^{(n+1)} \right)^2 \right] = \frac{1}{n+1} \cdot \frac{1}{\tau_x}. \)

The tree position of each trader becomes \( x_0^{(n+1)} \), i.e. the ex post efficient allocation perfectly smooths the tree positions in a local match. In other words, all the gains from trades are exhausted among \(n + 1\) traders. The drawback of this allocation is that it is too demanding in terms of the amount of information that each trader (and the planner) must know to implement the allocation. In fact, the allocation (4) is not incentive compatible. To see why it is not, note that (4) depends only on the initial tree positions \(x_0\) and is independent of signals \(s_0\), while the utility (5) depends on initial tree positions and signals. To achieve (5), signals must be shared truthfully. To induce truthful report of signals, the money allocation should not depend on signals. However, that would make it impossible to induce truthful report of the tree endowments, because a trader with a high signal has an incentive to receive more trees, therefore to report lower than actual tree endowments. We can restate this incentive problem in terms of information aggregation: although the allocation (4) aggregates no information about signals, it was derived based on the beliefs in (3) which require information about signals. In other words, there is an inconsistency between the informational requirement for the ex post efficient allocation and the informational consequence of the ex post efficient allocation. A natural question is whether there is an alternative notion of efficient allocation whose informational requirement and consequence are consistent. In the next subsection, we introduce such an efficiency criteria.
3.2 Post-trade efficient allocation

To develop a notion of efficiency that is consistent with information aggregation, we ask the following question: for a given environment, what is the minimum amount of information that an allocation can reveal to each trader such that the allocation is efficient with respect to the revealed information? We formally define an allocation that has this property, and call it a post-trade efficient allocation. In terms of its informational requirement, the post-trade efficiency is less demanding than the ex post efficiency, while it maintains a property that gains from trades are exhausted after the trade.

**Definition** A post-trade efficient allocation \( q^{pt} = \{q^{pt}_i\}_{i=1}^{n+1} \) is a solution to

\[
q^{pt} = \arg \max_q \sum_{i=1}^{n+1} \left( E_i [\tau_i | s_0, x_{i0}, q^{pt}] - \frac{\rho}{2} (q_i + x_{i0})^2 \right)
\]

subject to \( \sum_{i=1}^{n+1} q_i = 0. \) (6)

The defining feature of the post-trade efficiency is \( q^{pt} = q \) in the expectation operator: the allocation rule \( q^{pt} \) is used to evaluate the expected utility.\(^4\) Unlike the ex post efficient allocation, \( q^{pt} \) is a solution to the fixed point problem defined by (6). Since the allocation is chosen taking its informational content into account, it is efficient even after the allocation is made locally public among \( n+1 \) traders. This definition shows that the post-trade efficiency lies between the ex post efficiency and the interim efficiency. For example, if a solution \( q^{pt} \) to (6) reveals \( \pi_0^{(n+1)} \), then it also satisfies the ex post efficiency (3). On the other hand, if a solution \( q^{pt} \) to (6) does not reveal any information, then it satisfies the interim efficiency.\(^5\)

\(^4\) We could define an alternative notion by replacing \( q^{pt} = q \) with \( q^{pt}_i = q_i \). The allocation we characterize below satisfy both definitions.

\(^5\) In other words, post-trade efficiency defined here is just one candidate of many potential efficiency criteria which lie between the interim efficiency and the ex post efficiency. However, we believe this is a natural one to consider.
However, unlike the interim and ex-post efficiency, information structure must be solved, since the information revealed by $q^{pt}$ cannot be specified a priori. In fact, it is easy to verify that the ex post efficient allocation (4) does not satisfy (6).

To explicitly solve for a post-trade efficient allocation, note that any solution to (6) must equalize marginal valuations of traders. Since marginal valuations depend on beliefs that are implicit in (6), we must start with some conjecture. We conjecture that trader $i$’ marginal valuations take the following linear form:

$$MV_i(q_i; s_{i0}, x_{i0}) = \beta_s s_{i0} - \beta_x x_{i0} - \beta_q q_i.$$ (7)

Equating marginal valuations for any $i \neq j$,

$$MV_i(q_i; s_{i0}, x_{i0}) = MV_j(q_j; s_{j0}, x_{j0}) \iff \frac{MV_i + \beta_q q_i}{\beta_s} = \frac{MV_j + \beta_q q_j}{\beta_s}$$

$$\iff s_{i0} - \frac{\beta_x}{\beta_s} x_{i0} - \frac{\beta_q}{\beta_s} (q_i - q_j) = s_{j0} - \frac{\beta_x}{\beta_s} x_{j0}.$$  

Using $h_s(a, b) \equiv a - \frac{\beta_x}{\beta_s} b$ and $h_q(a, b) \equiv \frac{\beta_q}{\beta_s} (a - b)$, the condition above can be written as

$$h_s(s_{i0}, x_{i0}) - h_s(s_{j0}, x_{j0}) = h_q(q_i, q_j), \forall i \neq j.$$  

Therefore, for trader $i$, $\{q_i, q_j\}_{j \neq i}$ is informationally equivalent to $n$ independent signals

$$h_{ij}(s_{i0}, x_{i0}, q_i, q_j) \equiv h_s(s_{i0}, x_{i0}) - h_q(q_i, q_j)$$

$$= h_s(s_{j0}, x_{j0}) = s_{j0} - \frac{\beta_x}{\beta_s} x_{j0}$$

$$= v_A + \varepsilon_j - \frac{\beta_x}{\beta_s} x_{j0}, \forall j \neq i.$$ (8)

To see the informational contents of $h_{ij}$, by Bayes rule,

$$(Var[h_{ij}|v_A])^{-1} = \tau \varepsilon \varphi,$$
where \( \varphi \equiv \left\{ 1 + \left( \frac{\beta_x}{\beta_s} \right)^2 \frac{\tau_x}{\tau_s} \right\}^{-1} \) measures the share of information trader \( i \) can learn from observing \( h_{ij} \) relative to directly observing \( s_{j0} \). We use this \( \varphi \) as a measure of information aggregation. Note that \( \varphi \) depends on \( \frac{\beta_x}{\beta_s} \). It is determined as a level of information aggregation consistent with the marginal value equalization conditional on observation of trades \( \{q_i\}_{i=1}^{n+1} \).

Let \( \tau^{pt} \equiv \tau_v + \tau_\varepsilon (1 + n\varphi) \). Conditional on \( q \),

\[
E[v_A | s_{i0}, x_{i0}, q] = \frac{\tau_\varepsilon}{\tau^{pt}} \left\{ s_{i0} + \sum_{j \neq i} \varphi h_{ij}(s_{i0}, x_{i0}, q, j) \right\} \\
= \frac{\tau_\varepsilon}{\tau^{pt}} \left\{ s_{i0} + \varphi \sum_{j \neq i} (h_s(s_{i0}, x_{i0}) - h_q(q, j)) \right\} \\
= \frac{\tau_\varepsilon}{\tau^{pt}} \left\{ s_{i0} + \varphi \left( n h_s(s_{i0}, x_{i0}) - \sum_{j \neq i} \frac{\beta_q}{\beta_s} (q_i - q_j) \right) \right\} \\
= \frac{\tau_\varepsilon}{\tau^{pt}} \left\{ s_{i0} + n\varphi \left( s_{i0} - \frac{\beta_x}{\beta_s} x_{i0} \right) - \frac{\beta_q}{\beta_s} \left( nq_i - \sum_{j \neq i} q_j \right) \right\} \\
= \frac{\tau_\varepsilon}{\tau^{pt}} \left\{ (1 + n\varphi) s_{i0} - n\varphi \frac{\beta_x}{\beta_s} x_{i0} - (n + 1) \frac{\beta_q}{\beta_s} q_i \right\}.
\]

where the last equality used the feasibility condition \( q_i + \sum_{j \neq i} q_j = 0 \). Notice that this is linear in \((s_{i0}, x_{i0}, q_i)\). Thus, trader \( i \)'s marginal valuation conditional on \( q \) is

\[
E[v | s_{i0}, x_{i0}, q] - \rho(q_i + x_{i0}) \\
= \sqrt{u}E[v_A | s_{i0}, x_{i0}, q] - \rho(q_i + x_{i0}) \\
= \frac{\sqrt{u}\tau_\varepsilon}{\tau^{pt}} (1 + n\varphi) s_{i0} - \left( \sqrt{un}\varphi \frac{\beta_x}{\beta_s} + \rho \right) x_{i0} - \left( (n + 1) \varphi \frac{\beta_q}{\beta_s} + \rho \right) q_i.
\]

Solving a fixed point problem defined by (7) and (9) yields

\[
\beta_s = \frac{\sqrt{u} (1 + n\varphi^*)}{\tau_\varepsilon + 1 + n\varphi^*}, \quad \beta_x = \rho (1 + n\varphi^*), \quad \beta_q = \frac{1 + n\varphi^*}{1 - \varphi^*},
\]
where $\varphi^* \in (0, 1)$ is a unique solution to a cubic equation

$$\frac{\varphi}{1 - \varphi} = \frac{\frac{u \tau_e}{\rho^2 \tau_e} \cdot \frac{1}{(\frac{\tau_e}{\tau_e} + 1 + n \varphi)^2}}{1}.$$  \hspace{1cm} (10)

Note that both $\beta_x$ and $\beta_q$ would be $\rho$ in the absence of information aggregation ($\varphi^* = 0$), but they are greater than $\rho$ as long as $\varphi^* > 0$. Marginal valuations can now be written as

$$MV_i(q_i; s_{i0}, x_{i0}) = (1 + n \varphi^*) \left( \frac{\sqrt{u \tau_e \tau_{pt}}}{s_{i0} - \rho x_{i0} - \frac{\rho}{1 - \varphi^*} q_i} \right).$$  \hspace{1cm} (11)

To equate marginal valuations, (11) must be constant across traders. Hence, taking average across $n + 1$ traders and using the feasibility condition,

$$\frac{\sqrt{u \tau_e \tau_{pt}}}{s_{i0} - \rho x_{i0} - \frac{\rho}{1 - \varphi^*} q_i} = \frac{\sqrt{u \tau_e \tau_{pt}}}{s_{i0} - \rho x_{i0} - \frac{\rho}{1 - \varphi^*} q_i}$$

for all $i = 1, \ldots, n + 1$.

Hence,

$$q_{i_{pt}} = \frac{1 - \varphi^*}{\rho} \left\{ \frac{\sqrt{u \tau_e \tau_{pt}}}{s_{i0} - \frac{s_{0}^{(n+1)}}{x_{i0} - \frac{x_{0}^{(n+1)}}} - \rho \left( x_{i0} - \frac{x_{0}^{(n+1)}}{x_{0}^{(n+1)}} \right) \right\}$$  \hspace{1cm} (12)

The next lemma summarizes the analysis of the post-trade efficient allocation.

**Lemma 2**

(a) The post-trade efficient allocation among $n + 1$ traders is

$$q_{i_{pt}} + x_{i0} = (1 - \varphi^*) \left\{ \frac{\sqrt{u \tau_e \tau_{pt}}}{s_{i0} - \frac{s_{0}^{(n+1)}}{x_{i0} - \frac{x_{0}^{(n+1)}}} + \frac{x_{0}^{(n+1)}}{x_{0}^{(n+1)}}} + \varphi^* x_{i0}, \right\}$$  \hspace{1cm} (13)
and the associated utility is

\[
\left\{ \sqrt{u} \frac{(1 - \varphi^*)}{\tau + 1 + n \varphi^*} s_{i0} + \varphi^* \left( \rho x_{i0} + (n + 1) \left( \frac{\sqrt{u} S_0^{(n+1)}}{\tau + 1 + n \varphi^*} - \frac{\rho \pi_0^{(n+1)}}{\tau + 1 + n \varphi^*} \right) \right) \right\} (q^*_{i} + x_{i0})(14)
\]

where \( \varphi^* \in \left( \frac{u \tau_x}{u \tau_x + \tau \rho^2 \left( \frac{\tau_x}{\tau_x + 1 + n} \right)^2}, \frac{u \tau_x}{u \tau_x + \tau \rho^2 \left( \frac{\tau_x}{\tau_x + 1 + n} \right)^2} \right) \) is uniquely determined by (10).

(b) \( E \left[ (q^*_{i} + x_{i0})^2 \right] = (1 - \varphi^*) \sqrt{\frac{2}{\pi} n \left( \frac{1}{\tau_x} + \frac{u}{\rho^2 \tau_x \left( \frac{\tau_x}{\tau_x + 1 + n \varphi^*} \right)^2} \right)} \) and \( E \left[ (q^*_{i} + x_{i0})^2 \right] \)

and \( E \left[ (q^*_{i} + x_{i0})^2 \right] = \frac{1 + n \varphi^2}{n + 1} \frac{1}{\tau_x} + (1 - \varphi^*) \right) \frac{2}{n + 1} \frac{u}{\rho^2 \tau_x \left( \frac{\tau_x}{\tau_x + 1 + n \varphi^*} \right)^2} \).

(c) \( \frac{\partial \varphi^*}{\partial \tau_x} < 0, \quad \lim_{\tau_x \to 0} \frac{\rho^2}{u \tau_x} \varphi^* = \frac{1}{\tau_x \left( \frac{\tau_x}{\tau_x + 1 + n} \right)^2} \) and \( \lim_{\rho^2 \to \infty} \frac{\rho^2}{u \tau_x} \varphi^* = \frac{1}{\tau_x \left( \frac{\tau_x}{\tau_x + 1 + n} \right)^2} \).

(d) \( \frac{\partial \varphi^*}{\partial \tau_v} < 0, \quad \lim_{\tau_v \to \infty} \frac{\rho^2}{u \tau_x} \varphi^* = \frac{1}{\tau_x \left( \frac{\tau_x}{\tau_x + 1 + n} \right)^2} \).

(e1) If \( \tau_v = 0 \), then \( \lim_{\tau_x \to 0} \frac{\rho^2}{u \tau_x} \varphi^* = \frac{1}{\tau_x \left( \frac{\tau_x}{\tau_x + 1 + n} \right)^2} \) and \( \lim_{\tau_x \to \infty} \frac{\rho^2}{u \tau_x} \varphi^* = \frac{1}{\tau_x \left( \frac{\tau_x}{\tau_x + 1 + n} \right)^2} \).

(e2) If \( \tau_v > 0 \), then \( \exists \tau^*_v \in (0, \tau_v) \) s.t. \( \frac{\partial \varphi^*}{\partial \tau_v} \leq 0 \) \( \implies \tau^*_v \leq \tau^*_v \).

(f) \( \lim_{n \to \infty} n^2 \varphi^* = \frac{u \tau_x}{\rho \tau^*_v} \).

First, note that \( \frac{\sqrt{u} (s_{i0} - \tau_0^{(n+1)})}{\rho^2 \tau_x \left( \frac{\tau_x}{\tau_x + 1 + n \varphi^*} \right)^2} \) and \( \frac{\sqrt{u} \pi_0^{(n+1)}}{\rho^2 \tau_x \left( \frac{\tau_x}{\tau_x + 1 + n \varphi^*} \right)^2} - \rho \pi_0^{(n+1)} \) in (13) and (14) are informally equivalent for trader \( i \). Thus, it is just enough to know the allocation to form the associated belief. Importantly, \( \text{Lemma 2(b)} \) characterizes the trading volume and the dispersion of the allocation. These expressions show that, for a given \( n \), if \( \varphi^* \) approaches one, then the post-trade efficient outcome approaches a no-trade outcome given that \( \frac{u(1 - \varphi^*)^2}{\rho^2 \tau_x \left( \frac{\tau_x}{\tau_x + 1 + n \varphi^*} \right)^2} \) approaches zero. On the other hand, if \( \varphi^* \) approaches zero, then the post-trade efficient outcome approaches the ex-post efficient outcome, given that \( \frac{u}{\rho^2 \tau_x \left( \frac{\tau_x}{\tau_x + 1 + n} \right)^2} \) approaches zero. This shows that the degree of information aggregation \( \varphi^* \) is connected to a gap between the ex post efficient allocation and the post-trade efficient allocation. Importantly, high \( \varphi^* \) implies little trading and that the allocation is not smoothed. \( \text{Lemma 2(c-e)} \) shows various limiting cases. In particular, two cases \( u \to 0 \) and \( \rho \to \infty \) achieve \( \varphi^* \to 0 \).
and the ex post efficiency. Lemma 2(e) shows that the information aggregation $\varphi^*$ is non-monotonic in the precision of signals $\tau_\varepsilon$ if $\tau_v > 0$. Both limits $\tau_\varepsilon \to 0$ and $\tau_\varepsilon \to \infty$ achieve $\varphi^* \to 0$ and the ex post efficiency, while $\varphi^*$ takes the maximum value for $\tau_\varepsilon = \tau^*_\varepsilon \in (0, \tau_v)$.

Finally, Lemma 2(f) shows that increasing the size of local trading makes the post-trade efficient outcome closer to the ex post efficient outcome. However, the speed of convergence is slow: $\varphi^*$ is decreasing in $n$ at the rate of $n^{-\frac{2}{3}}$, and hence $n \varphi^*$ is increasing in $n$ only at the rate $n^{\frac{1}{3}}$. Thus, even though having more traders in one match improves efficiency in principle, it may be too costly in practice if 1,000 traders have an effective impact of 10 traders in terms of convergence. This also motivates us to study whether dynamic local trading can achieve the convergence.

Now that we established a relevant efficiency benchmark, we move on to study a local trading rule in the next section.

4 Trading rule

A trading rule we study is a variation of trading rule studied in Kyle (1989). There are two reasons for our choice of the rule. First, we focus on a trading rule which aggregates the same amount of information that would be aggregated in the post-trade efficient allocation. Because different trading rules generally change both an allocation and an information structure in our environment, this allows us to compare two allocations by keeping the same endogenous information structure. Second, we use a trading rule that induces a price-taking behavior to keep the dynamic analysis tractable. To do this, we need to use money allocations $\{r_i\}_{i=1}^{n+1}$. Trader $i$'s money trade is determined by

$$r_i = -p_i q_i,$$

\footnote{Bernhardt, Seiler and Taub (2010) and Rostek and Weretka (2011) also present a dynamic analysis of demand-submission markets.}
where $p_i$ is the unit price of the tree for trader $i$. Because we do not impose any restriction on the money position, the money endowment $\{b_i\}_{i=1}^{n+1}$ is not important for our analysis. An individual price $p_i$ is determined by the following pricing rule

$$\sum_{j \neq i} q_j (p_i) = 0,$$  \hfill (15)

where $\{q_i(\cdot)\}_{i=1}^{n+1}$ is price-contingent orders submitted by traders. Hence, trader $i$’s unit price $p_i$ is determined independent of his order $q_i(\cdot)$, while his tree allocation is determined by $q_i(p_i)$. This makes each trader a price-taker, but internalize informational contents of prices by best-responding to every possible realization of $p_i$. Due to the ex ante symmetry among traders, it turns out that prices $\{p_i\}_{i=1}^{n+1}$ determined by (15) satisfy the local market clearing

$$\sum_{i=1}^{n+1} q_i (p_i) = 0$$

in a symmetric equilibrium we study.

This rule captures the idea that a market maker can offer different prices for different traders to “locally clear” a market, while allowing traders to adjust the quantity traded at the offered prices. Although we do not model intermediaries explicitly, we believe that it is reasonable to assume that prices are formed individually rather than uniformly in many trading situations. One way to interpret the pricing rule (15) is that, the intermediary offers trader $i$ a hypothetical price at which the other $n$ traders would happily trade the tree among them. The intermediary can do this because he has the orders from the other traders. This price offer provides a useful information to trader $i$, which he internalizes in his order. Because every trader does the same reasoning, there will be information aggregation in equilibrium. In this sense, this trading rule captures the informational advantage of intermediaries NOT because of the intrinsic superior knowledge about the tree, but because of the orders they receive.

Before studying dynamic trading in the next section, we compare a static equilibrium allocation of this trading game with the post-trade efficient allocation. To characterize the
equilibrium allocation, conjecture the order of the form:

\[ q_i(p; s_{i0}, x_{i0}) = \beta_s s_{i0} - \beta_x x_{i0} - \beta_p p. \]

The characterization proceeds similarly as before, and details are gathered in the appendix. A key feature of this trading rule is that prices take the following form

\[ \frac{\beta_p}{\beta_s} p_i = \frac{1}{n} \sum_{j \neq i} \left( s_{j0} - \frac{\beta_x}{\beta_s} x_{j0} \right). \quad (16) \]

Comparing (8) and (16), it is clear that prices aggregate the same amount of information as in the post-trade efficient allocation if and only if \( \frac{\beta_p}{\beta_s} \) takes the same value in both cases. Indeed, we show that it is the case: each trader’s order is informationally equivalent to a noised-up signal \( h_i \equiv s_{i0} - \frac{\beta_x}{\beta_s} x_{i0} = s_{i0} - \frac{\rho \tau}{\sqrt{w \epsilon}} x_{i0}, \) which is informationally equivalent to the signal, \( h_s(s_{i0}, x_{i0}) = s_{i0} - \frac{\beta_x}{\beta_s} x_{i0}, \) revealed in the post-trade efficient allocation. We summarize the static results below. We use the notation \( \overline{s}_{0,-i} \equiv \frac{1}{n} \sum_{j \neq i} s_{j0} \) etc to denote the average except trader \( i. \)

**Lemma 3**

(a) \( q_i^*(p_i) = \frac{1+n}{n(1-\varphi^*)} q_i^{pt}, \) and the trading allocation among \( n+1 \) traders is

\[ q_i^*(p_i) + x_{i0} = \frac{\sqrt{u \tau}}{\rho \tau^{pt}} \left( s_{i0} - \overline{s}_{0,-i} \right) + \overline{x}_{0,-i} \]

\[ = \frac{n+1}{n} \left\{ \frac{\sqrt{u}}{\rho \left( \frac{\tau}{\epsilon} + 1 + n \varphi^* \right)} \left( s_{i0} - \overline{s}_{0,-i} \right) + \overline{x}_{0,-i}^{(n+1)} \right\} - \frac{1}{n} x_{i0}, \]

where \( \varphi^* \) is uniquely determined by (10).

(b) \( \frac{1+n}{n(1-\varphi^*)} > \frac{n+1}{n} \). For \( a > \frac{n+1}{n}, \frac{1+n}{n(1-\varphi^*)} \approx a \) if and only if \( \varphi^* \leq \frac{1}{a} \left( a - \frac{n+1}{n} \right). \)

(c) \( \sum_{i=1}^{n+1} r_i^* = - \sum_{i=1}^{n+1} p_i q_i^*(p_i) = \frac{u(1+n \varphi^*)}{\rho \left( \frac{\tau}{\epsilon} + 1 + n \varphi^* \right)} \sum_{i=1}^{n+1} (h_i - h_{-i})^2 > 0, \)

and \( E \left[ \sum_{i=1}^{n+1} r_i^* \right] = (n+1) \frac{u}{\rho \tau^{pt}} \left( \frac{\tau}{\epsilon} + 1 + n \varphi^* \right) \varphi^*. \)
Lemma 3 shows that this trading rule induces too much trading relative to the post-trade efficiency, while the amount of information aggregation is same across two allocations. The amount of excessive trading is larger for smaller $n$, because $\varphi^*$ is decreasing in $n$ from Lemma 2 and hence $\frac{1+n}{n(1-\varphi^*)}$ is decreasing in $n$. In particular, when $n = 1$, the trading volume induced by this rule is at least twice as big as the post-trade efficiency benchmark. As $n$ increases, this gap converges to zero, and both allocations converge to the ex post efficient allocation. For a given $n$, the deviation from the efficiency benchmark becomes larger as more information is aggregated. Thus, the model suggests that information aggregation creates more excess trading relative to the post-trade efficiency benchmark. Lemma 3(d) shows that the expected average subsidy is positive and increasing in $n$ without bound. For the rest of the paper, we maintain the assumption of small fixed $n$. Given that increasing the size of local interaction requires a larger subsidy in our environment, a natural question is whether dynamic trading can achieve convergence to the ex post efficient allocation while keeping $n$ fixed. In the next section, we investigate how information aggregation and trading interact in a dynamic model.

5 Dynamic trading

We denote each trader’s tree trade at time $t$ by $q_{it}$ and let $r_{it} = -p_{it}q_{it}$ be the associated money trade. A trader $i$’s payoff at the end of time $t$ is $\pi_{it} = v\left(\sum_{s=1}^{t} q_{is} + x_{i0}\right) + \sum_{s=1}^{t} r_{is} + b_{i0}$.

At the end of each trading round $t \leq T - 1$, there is positive probability $1 - \gamma \in (0, 1)$ that the game ends and no more trading is allowed. If the game ends after trading at time $t$, the trader $i$’s expected utility is

$$U_{it} = E[v|\mathcal{F}_{it}] \left(\sum_{s=1}^{t} q_{is} + x_{i0}\right) + \sum_{s=1}^{t} r_{is} + b_{i0} - \frac{\rho}{2} \left(\sum_{s=1}^{t} q_{is} + x_{i0}\right)^2 .$$ (17)
The information set $\mathcal{F}_{it}$ expands over time reflecting the new information that trader $i$ learns from trading over time. The expected lifetime utility evaluated at time $t$ is defined by

$$V_{it} \equiv \frac{1 - \gamma}{1 - \gamma^{T-t+1}} E \left[ \sum_{s=t}^{T} \gamma^{s-t} U_{is} | \mathcal{F}_{it} \right]$$

$$= \begin{cases} 
(1 - \gamma) U_{it} + \gamma E[V_{it+1} | \mathcal{F}_{it}] & \text{for } t \leq T - 1 \\
U_{iT} & \text{for } t = T 
\end{cases}.$$  

(18)

The parameter $\gamma$ is a probability weight put on future trading rounds, and measures how forward-looking traders are. If $\gamma = 0$, traders are assumed to be myopic in that they care only period payoff (17) even though trading rounds continue and information is accumulated over time. An alternative model interpretation is that $1 - \gamma$ is a probability of aggregate event that forces traders to consume their positions.

The trading at each period is locally intermediated as described in section 4. At each time $t = 1, ..., T$, given that the game has not ended, each trader is matched with a finite number $n \in \mathbb{N}$ of other traders randomly drawn from the population. Each trader submits his order $q_{it}(\cdot; \mathcal{F}_{it})$ to a market maker, which is explicitly conditioned on the unit price $p_{it}$ he must pay. Given submitted $n + 1$ orders $\{q_{it}(\cdot; \mathcal{F}_{it})\}_{i=1}^{n+1}$, the market maker determines $\{p_{it}\}_{i=1, ..., n+1}$ subject to the constraint

$$\sum_{i=1}^{n+1} q_{it}(p_{it}; \mathcal{F}_{jt}) = 0.$$  

(19)

Each trader $i$ is charged $p_{it}$ which satisfies

$$\sum_{j \neq i} q_{jt}(p_{it}; \mathcal{F}_{jt}) = 0.$$  

(20)

That is, the trader $i$ pays the customized unit price $p_{it}$ to the market maker. The price $\{p_{it}\}_{i=1}^{n+1}$ is constructed as a hypothetical market-clearing price for the $n$ orders from the other traders. The tree is allocated according to $\{q_{it}(p_{it}; \mathcal{F}_{it})\}_{i=1}^{n+1}$. The monetary payment
is \( r_{it} = \rho_{it} q_{it}(p_{it}; \mathcal{F}_{it}) \).

In the next trading round, each trader is randomly matched with another \( n \) traders. We assume that no trader is matched with the same trader more than once regardless of the size of local trading \( n + 1 \).

**Definition** A dynamic equilibrium is a collection \( \{q_{it} (\cdot; \mathcal{F}_{it}), p_{it}, \mathcal{F}_{it}\}_{i \in I, t = 1, \ldots, T} \) which satisfy for \( t = 1, \ldots, T \),

(i) For all \( i \in I \), \( q_{it} (\cdot; \mathcal{F}_{it}) = \arg \max_{q_{it}} \mathcal{V}_{it} \), where \( \mathcal{F}_{it} = \mathcal{F}_{it-1} \cup \{p_{it}, x_{it-1}\} \)
and \( \mathcal{F}_{i0} = \{s_{i0}, x_{i0}\} \).

(ii) For each local trading, (20) determines \( p_{it} \) for \( i = 1, \ldots, n + 1 \).

(iii) Each trader forms a Bayesian belief about the distributions of \( v \) and \( \{p_{it}, x_{it-1}\}_{i \in I, t = 1, \ldots, T} \)
consistent with the other equilibrium variables.

In the model, there is once and for all exogenous information arrival at time 0. At time \( t \), each trader’s information set \( \mathcal{F}_{it} \) contains the initial private information \((s_{i0}, x_{i0})\) and other information obtained in the past trading rounds \( s \leq t - 1 \). In the equilibrium definition above, the information set at time \( t \) also contains the price he pays at time \( t \) (i.e., \( p_{it} \in \mathcal{F}_{it} \))
even though each trader does not know the realization of \( p_{it} \) when he submits his order at time \( t \). Since the order at time \( t \) is allowed to be conditioned on \( p_{it} \), each trader can choose his best response for each realization of \( p_{it} \).

We did not include the zero profit condition for the market-making sector in the definition above because this trading rule requires subsidy on average. Market makers would have to charge a positive fixed fee if they were required to break even in each trading round. This does not change the trading behavior, but may affect traders’ participation decision depending on their information state.\(^7\) To keep the trading dynamics as simple as possible, we assume that the market making sector subsidizes traders. Alternatively we could assume that traders

\(^7\)With a positive fee for each trading round, there will be a positive measure of traders whose tree positions happen to be so close to perfect risk sharing that they do not want to pay the fee to trade. This exit behavior makes the distribution of tree positions truncated normal, and makes the analysis intractible.
can commit to the fixed fee payment ex ante, before they know their private information. Competition among market makers will bid down the fee level so that they break even in expectation.

5.1 No information aggregation

To demonstrate that our trading rule is a reasonable one, we study a case \( u = 0 \), i.e., with symmetric information about the tree. In this case, \( E[v|\mathcal{F}_{it}] = E[v_B|\mathcal{F}_{it}] = 0 \) for all \( t \) and \( i \). The ex post efficient allocation among \( n + 1 \) traders is still \( \pi_0^{(n+1)} \). We show that dynamic trading approaches the ex post efficient allocation given that \( n \geq 2 \) and there are sufficient numbers of trading rounds.

**Lemma 4** In a dynamic equilibrium without private signals,

\[
q_{it}^* (p_{it}^*; x_{it-1}) = \bar{x}_{t-1,-i} - x_{it-1}.
\]

*Conditional on the game not ending, the tree position at the end of time \( t \) is*

\[
x_{it} = \frac{1}{n_t} \sum_{j=1}^{n_t} x_{jt0},
\]

*and the cross-sectional distribution of the tree is \( N \left( 0, n^{-t} \frac{1}{\tau_x} \right) \).*

For every trading round \( t \), trader \( i \) is matched with \( n \) new traders. In our random-matching environment with a continuum of traders, a group of traders that have traded together will never meet again. Also, their sets of trading counterparties do not overlap. Thus, by the end of time \( t \), trader \( i \) has traded with \( nt \) other traders, but indirectly exchanged their positions with \( n^t \) traders. Therefore, position of the tree converges at the rate \( n^t \). Compared with the ex post efficient allocation \( \pi_{t-1}^{(n+1)} \), one round of trading achieves less smoothing than the ex post efficient allocation, since \( \pi_{t-1,-i} \) is the average of \( n \) traders’
positions, not \( n + 1 \). However, as long as \( n \geq 2 \), dynamic trading can achieve the ex post efficient allocation after the sufficient number of trading rounds.\(^8\)

### 5.2 Dynamic information aggregation

To analyze the interaction of dynamic information aggregation and allocation, we construct a dynamic equilibrium with the following properties:

Property I: At the beginning of period \( t \), trader \( i \) has the tree position \( x_{it-1} \), and the cross-sectional distribution of \( x_{it-1} \) is \( N \left( 0, \tau_{xt-1}^{-1} \right) \).

Property II: At the beginning of period \( t \), trader \( i \) has \( t \) signals \( \{s_{ik}\}_{k=0}^{t-1} \), where \( s_{ik} = u_A + \varepsilon_{ik} \), \( k = 0, \ldots, t - 1 \), and the distribution of the noise \( \varepsilon_{ik} \) is \( N \left( 0, \frac{1}{\tau_{xt}^{k} \Phi_k} \right) \) independent across traders.

Property III: For time \( t \) trading, trader \( i \) submits an order

\[
q_{it}(p_{it}; F_{it}) = \sum_{k=0}^{t-1} \beta_{stk}s_{ik} + \beta_{xt}x_{it-1} - \beta_{pt}p_{it}.
\]

Note that Property I and II are satisfied at \( t = 1 \) with \( \phi_0 \equiv 1 \), given our assumptions on \( \{s_{i0}, x_{i0}\}_{i \in I} \). By the end of time \( t \) trading, each trader has directly interacted with \( nt \) other traders. However, because no trader meets twice, each trader has indirectly interacted with \( n^t \) other traders. Therefore, the most trader \( i \) can potentially learn from time \( t \) trading is the information that would be obtained from \( n^t \) independent signals \( \{s_{j0}\} \). Thus, \( \phi_k \) in Property II measures a fraction of information each trader learns at time \( k \) relative to the potentially available information \( \tau_n^k \). For each \( t = 1, \ldots, T \), conjectured \( t + 2 \) coefficients \( \{\beta_{stk}\}_{k=0}^{t-1}, \beta_{xt}, \beta_{pt}\) must be verified and characterized. Also, signals \( \{s_{ik}\}_{k=1}^{t-1} \) must be constructed from the equilibrium trading behavior to satisfy Property II.

---

\(^8\)When each local trading is bilateral (\( n = 1 \)), this convergence does not occur, because two traders switch their tree positions: \( x_{t-1, -i} = x_{jt-1} \).
Given the pricing rule (20) and the conjecture (21), information learned from \( p_t \) is

\[
s_{it} \equiv \frac{\beta_{pt}}{\sum_{k=0}^{t-1} \beta_{st}} p_{it} = \frac{1}{n} \sum_{j \neq i=0}^{t-1} \frac{\sum_{k=0}^{t-1} \beta_{st} s_{jk}}{\sum_{k=0}^{t-1} \beta_{st}} - \frac{\beta_{xt}}{\sum_{k=0}^{t-1} \beta_{st}} x_{t-1,-i}. \]

Using \( \tilde{\beta}_{st} \equiv \frac{\beta_{st}}{\sum_{k=0}^{t-1} \beta_{st}} \) and \( \tilde{\beta}_{xt} \equiv \frac{\beta_{xt}}{\sum_{k=0}^{t-1} \beta_{st}} \), this can be written as

\[
s_{it} \equiv \tilde{\beta}_{pt} p_{it} = \sum_{k=0}^{t-1} \tilde{\beta}_{st} s_{k,-i} - \tilde{\beta}_{xt} x_{t-1,-i}.
\]

Suppose Property II holds at period \( t \). Then the signal \( s_{it} \) is independent across \( i \) and also independent from \( \{s_{ik}\}_{k=0}^{t-1} \) conditional on \( v_A \), because no pair of traders meets twice and does not share the history before matching. Therefore,

\[
(Var[s_{it}|v_I])^{-1} = \left( \sum_{k=0}^{t-1} \tilde{\beta}_{st}^2 \frac{1}{n} \frac{1}{\tau x_{k}^{\varphi_k}} + \tilde{\beta}_{xt}^2 \frac{1}{n} \frac{1}{\tau x_{t-1}} \right)^{-1}.
\]

Then we can define \( \varphi_t \) by

\[
\frac{1}{n^{t-1} \varphi_t} = \sum_{k=0}^{t-1} \tilde{\beta}_{st}^2 \frac{1}{n} \frac{1}{\tau x_{k}^{\varphi_k}} + \tilde{\beta}_{xt}^2 \frac{1}{n} \frac{1}{\tau x_{t-1}}.
\]

Thus, Property II holds at period \( t + 1 \) with \( s_{it} \) defined by (22) and \( \varphi_t \) defined by (23).

Note that trader \( i \)'s accumulated signals \( \{s_{ik}\}_{k=1}^{t-1} \) are informationally equivalent to his price history \( \{p_{ik}\}_{k=1}^{t-1} \).

Given the conjecture (21), prices and quantity traded in equilibrium at time \( t \) must satisfy

\[
\beta_{pt} p_{it}^e = \sum_{k=0}^{t-1} \tilde{\beta}_{st} s_{k,-i} - \tilde{\beta}_{xt} x_{t-1,-i}.
\]
\[ q_{it}(p^*_it; \mathcal{F}_{it}) = \sum_{k=0}^{t-1} \beta_{stk} (s_{ik} - \overline{s}_{k,-i}) - \beta_{xt} (x_{it-1} - \overline{x}_{t-1,-i}). \]

Hence, \( x_{it-1} \) and \( x_{it} \) are related in equilibrium by the following condition:

\[ x_{it} = \sum_{k=0}^{t-1} \beta_{stk} (s_{ik} - \overline{s}_{k,-i}) + \beta_{xt} \overline{x}_{t-1,-i} + (1 - \beta_{xt}) x_{it-1}. \quad (24) \]

Suppose Property I and II hold at time \( t \). Given that \( x_{it-1} \) has a distribution \( N(0, \tau_{xt-1}^{-1}) \) and that \( \varepsilon_{ik} \) has \( N\left(0, \frac{1}{\tau_{\varepsilon} n^k \varphi_k}\right) \) independent across \( i \) up to \( k = t - 1 \), (24) implies that \( E[x_{it}] = 0 \) and

\[
V[x_{it}] = \frac{1}{n} \left( \sum_{k=0}^{t-1} \frac{\beta_{stk}^2}{\tau_{\varepsilon} n^k \varphi_k} + \frac{\beta_{xt}^2}{\tau_{xt-1}} \right) + \sum_{k=0}^{t-1} \frac{\beta_{stk}^2}{\tau_{\varepsilon} n^k \varphi_k} + (1 - \beta_{xt})^2 \frac{1}{\tau_{xt-1}}
\]

\[
= \frac{1}{\tau_{\varepsilon} n^t \varphi_t} \left( \sum_{k=0}^{t-1} \beta_{stk} \right)^2 + \sum_{k=0}^{t-1} \frac{\beta_{stk}^2}{\tau_{\varepsilon} n^k \varphi_k} + (1 - \beta_{xt})^2 \frac{1}{\tau_{xt-1}}.
\]

Therefore,

\[
\frac{\tau_{\varepsilon}}{\tau_{xt}} = \left( \sum_{k=0}^{t-1} \beta_{stk} \right)^2 \left( \frac{1}{n^t \varphi_t} + \sum_{k=0}^{t-1} \frac{\beta_{stk}^2}{n^k \varphi_k} \right) + (1 - \beta_{xt})^2 \frac{\tau_{\varepsilon}}{\tau_{xt-1}}. \quad (25)
\]

Thus, Property I holds at time \( t + 1 \) with \( \tau_{xt} \) determined by (25).

Two dynamic equations (23) and (25) jointly describe learning and allocation dynamics given equilibrium trading behavior characterized by \( \{ \beta_{stk} \}_{k=0}^{t-1} ; \beta_{xt} ; \beta_{pt} \}_{t=1}^{T} \). It should be clear from the derivation of these two equations that Properties I and II hold for time \( t \) if they hold for time up to \( t - 1 \). Hence, Properties I and II were verified by induction given Property III.

Next, we verify Property III by showing that trader \( i \)'s optimal order takes the conjectured form (21) given that all the others use the same form. Each trader \( i \)'s belief about \( v_A \) at time \( t \) is summarized by its conditional mean and variance, and characterized by Bayes rule.
To suppress expressions, let \( \chi_t \equiv \sum_{k=0}^{t} n^k \varphi_k \). Note that \( \chi_t \geq 1 \) because \( \varphi_0 \equiv 1 \).

\[
E_{it}[v_A] = \frac{\sum_{k=0}^{t} n^k \varphi_k s_{ik}}{\tau_x / \tau + \chi_t}
= \frac{1}{\tau_x / \tau + \chi_t} \left( \sum_{k=0}^{t-1} n^k \varphi_k s_{ik} + n^t \varphi_t \tilde{\beta}_{pt} p_{it} \right). 
\] (26)

Note that this is linear in \( \{s_{ik}\}_{k=0}^{t-1} \) and \( p_{it} \). Given this belief, we derive the optimal order. First, consider the final period \( t = T \). Because there is no more trading after period \( T \), the optimal order takes the same form as in the static case:

\[
q_{iT}(p_{iT}; \mathcal{F}_{iT}) = \frac{\sqrt{u} E_{iT}[v_A] - p_{iT} x_{iT-1}}{\rho - x_{iT-1} - \frac{1}{\rho} \left( 1 - \frac{\sqrt{u} n^T \varphi_T \tilde{\beta}_{pT}}{\tau_x / \tau + \chi_T} \right) p_{iT}.
\]

By equating coefficients with those in (21) for \( t = T \),

\[
\beta_{sTk} = \frac{\sqrt{u}}{\rho} \frac{n^k \varphi_k}{\tau_x / \tau + \chi_T}, \quad k = 0, \ldots, T - 1,
\]

\[
\beta_{xT} = 1 \quad \text{and} \quad \beta_{pT} = \frac{1}{\rho} \left( 1 - \frac{\sqrt{u} n^T \varphi_T \tilde{\beta}_{pT}}{\tau_x / \tau + \chi_T} \right).
\]

From the expression of \( \beta_{sTk} \),

\[
\sum_{k=0}^{T-1} \beta_{sTk} = \frac{\sqrt{u}}{\rho} \frac{\chi_{T-1}}{\tau_x / \tau + \chi_T} \quad \text{and} \quad \tilde{\beta}_{sTk} = \frac{n^k \varphi_k}{\chi_{T-1}}, \quad k = 0, \ldots, T - 1,
\]

\[
\tilde{\beta}_{xT} = \frac{\rho}{\sqrt{u}} \frac{\chi_T}{\chi_{T-1}},
\]

\[
\tilde{\beta}_{pT} = \frac{1}{\sqrt{u} \chi_{T-1}} \left( 1 - \frac{\sqrt{u} n^T \varphi_T \tilde{\beta}_{pT}}{\tau_x / \tau + \chi_T} \right)
= \frac{1}{\sqrt{u} \chi_{T-1}} \left( \frac{\tau_v}{\tau} + \chi_T - \sqrt{u} n^T \varphi_T \tilde{\beta}_{pT} \right).
\]

24
The last condition can be solved for \( \tilde{\beta}_{\mu_T} \):

\[
\tilde{\beta}_{\mu_T} = \frac{\frac{\tau_u}{\tau_\varepsilon} + \chi_T}{\sqrt{u} (\chi_{T-1} + n^T \varphi_T)} = \frac{\frac{\tau_u}{\tau_\varepsilon} + \chi_T}{\sqrt{u} \chi_T}.
\]

Hence,

\[
\beta_{\mu_T} = \frac{1}{\rho} \left( 1 - \frac{\sqrt{u} n^T \varphi_T (\frac{\tau_u}{\tau_\varepsilon} + \chi_T)}{\sqrt{u} \chi_T} \right) = \frac{1}{\rho} \frac{\chi_{T-1}}{\chi_T}.
\]

Hence, for \( t = T \), Property III was verified and

\[
q_{iT}(p_{iT}; F_{iT}) = \frac{\sqrt{u}}{\rho \left( \frac{\tau_u}{\tau_\varepsilon} + \chi_T \right)} \sum_{k=0}^{T-1} n^k \varphi_k s_{ik} - x_{iT-1} - \frac{1}{\rho} \frac{\chi_{T-1}}{\chi_T} p_{iT},
\]

\[
p_{iT}^* = \sum_{k=0}^{T-1} \frac{\beta_{stk}}{\beta_{pt}} \bar{s}_{k,-i} - \frac{\beta_{xt}}{\beta_{pt}} \bar{x}_{T-1,-i} \quad \text{(27)}
\]

\[
q_{iT}(p_{iT}^*; F_{iT}) = \frac{\sqrt{u}}{\rho \left( \frac{\tau_u}{\tau_\varepsilon} + \chi_T \right)} \sum_{k=0}^{T-1} n^k \varphi_k (s_{ik} - \bar{s}_{k,-i}) - (x_{iT-1} - \bar{x}_{T-1,-i} - \bar{x}_{T-1,-i}) \quad \text{(28)}
\]

\[
x_{iT} = \frac{\sqrt{u}}{\rho \left( \frac{\tau_u}{\tau_\varepsilon} + \chi_T \right)} \sum_{k=0}^{T-1} n^k \varphi_k (s_{ik} - \bar{s}_{k,-i}) + \bar{x}_{T-1,-i} \quad \text{(29)}
\]

Substituting derived coefficients into (23) gives

\[
\frac{1}{n^{T-1} \varphi_T} = \sum_{k=0}^{T-1} \left( \frac{n^k \varphi_k}{\chi_{T-1}} \right)^2 + \left( \frac{\rho}{\sqrt{u}} \frac{\tau_u}{\tau_\varepsilon} + \chi_T \right) \left( \frac{\tau_u}{\tau_\varepsilon} + \chi_T \right) \frac{\tau_\varepsilon}{\tau_{xT-1}} = \frac{1}{\chi_{T-1}} + \rho^2 \frac{\tau_\varepsilon}{u} \tau_{xT-1} \left( \frac{\tau_u}{\tau_\varepsilon} + \chi_T \right) \frac{\tau_\varepsilon}{\chi_{T-1}}.
\]
This can be seen as an equation in \( \varphi_T \) given \((\varphi_1, \ldots, \varphi_{T-1})\) and \(\tau_{xT-1}\):

\[
\frac{n_{T-1}^{-1} \varphi_T}{\chi_{T-1}} - 1 + \frac{\rho^2}{u} \frac{\tau_{\varepsilon}}{\tau_{xT-1}} \left( \frac{\tau_{\varepsilon} + \chi_T}{\chi_{T-1}} \right)^2 n_{T-1}^{-1} \varphi_T = 0.
\]

\[
\iff \frac{\rho^2}{u} \frac{\tau_{\varepsilon}}{\tau_{xT-1}} \left( \frac{\tau_{\varepsilon} + \chi_T}{\chi_{T-1}} \right)^2 n_{T-1}^{-1} \varphi_T = 1 - \frac{n_{T-1}^{-1}}{\chi_{T-1}} \varphi_T.
\]

Because the right hand side must be positive for the solution to exist,

\[
\frac{n_{T-1}^{-1} \varphi_T}{1 - \frac{n_{T-1}^{-1}}{\chi_{T-1}} \varphi_T} = \frac{u}{\rho^2} \frac{\tau_{xT-1}}{\tau_{\varepsilon}} \left( \frac{\chi_{T-1}}{\tau_{\varepsilon} + \chi_T} \right)^2.
\]

(30)

For a fixed \((\varphi_1, \ldots, \varphi_{T-1})\) and \(\tau_{xT-1}\), the left hand side of (30) is increasing in \(\varphi_T\) and continuously change from zero to positive infinity for \(\varphi_T \in \left[0, \frac{\chi_T^{-1}}{n_{T-1}}\right]\) and the right hand side is decreasing in \(\varphi_T\). Hence, there is a unique \(\varphi_T^*\) that solves (30) for any given \((\varphi_1, \ldots, \varphi_{T-1})\) and \(\tau_{xT-1}\).

Also, substituting derived coefficients into (25) gives

\[
\frac{\tau_{\varepsilon}}{\tau_{xT}} = \left( \frac{\sqrt{u}}{\rho} \frac{\chi_{T-1}}{\tau_{\varepsilon} + \chi_T} \right)^2 \left( \frac{1}{n^T \varphi_T} + \sum_{k=0}^{T-1} \frac{n^{k} \varphi_k}{n^T \varphi_T} \right) \left( \frac{1}{\chi_{T-1}} + \frac{1}{\chi_T} \right).
\]

\[
= \frac{u}{\rho^2} \left( \frac{\chi_{T-1}}{\tau_{\varepsilon} + \chi_T} \right)^2 \left( \frac{1}{n^T \varphi_T} + \frac{1}{\chi_{T-1}} \right) \chi_T.
\]

\[
= \frac{u}{\rho^2} \left( \frac{\chi_{T-1}}{\tau_{\varepsilon} + \chi_T} \right)^2 \frac{\chi_T}{n^T \varphi_T \chi_{T-1}}
\]

\[
= \frac{u}{\rho^2 n^T \varphi_T} \frac{\chi_{T-1} \chi_T}{\left( \frac{\tau_{\varepsilon}}{\tau_{xT}} + \chi_T \right)^2}.
\]

Because \(\beta_{xT} = 1\) under the current trading rule, whatever position traders have at the beginning of period \(T\) is traded away and averaged out and \(x_{iT-1}\) does not directly affect \(x_{iT}\). Hence \(\frac{\tau_{\varepsilon}}{\tau_{xT-1}}\) does not explicitly show up in the expression of \(\frac{\tau_{\varepsilon}}{\tau_{xT}}\) above. However, (30) shows that \(\varphi_T\) is decreasing in \(\frac{\tau_{\varepsilon}}{\tau_{xT-1}}\). Hence, the distribution of trees in the previous period
affects the distribution of the tree in the following period through the learning channel.

Next, we characterize an optimal order in period $T - 1$. Trader $i$ solves

$$
\max_{q_{iT-1}} (1 - \gamma) \left\{ E_{iT-1} [v] (q_{iT-1} + x_{iT-2}) - \frac{\rho}{2} (q_{iT-1} + x_{iT-2})^2 - p_{iT-1}q_{iT-1} \right\} \\
+ \gamma E_{iT-1} \left[ E_{iT} [x_{iT} - \frac{\rho}{2} x_{iT}^2 - (p_{iT-1}q_{iT-1} + p^*_iTq_{iT} (p^*_iT)) \right],
$$

where $p^*_iT$, $q_{iT}$ ($p^*_iT$), $x_{iT}$ are given by (27), (28), (29). The first line in the expression above is the expected utility for the case the game ends after trading in period $T - 1$, while the second line corresponds to the case where there is another trading round. First, money payment $p_{iT-1}q_{iT-1}$ is sunk and hence shows up in both lines. Second, given the optimal order in period $T$, neither $p^*_iT$ nor $x_{iT}$ depends on $q_{iT-1}$, while $q_{iT}$ ($p^*_iT$) does depend on $q_{iT-1}$ through the term $-x_{iT-1} = -(q_{iT-1} + x_{iT-2})$. Thus, dropping irrelevant terms, the objective can be written as

$$
\max_{q_{iT-1}} (1 - \gamma) \left\{ E_{iT-1} [v] q_{iT-1} - \frac{\rho}{2} (q_{iT-1} + x_{iT-2})^2 \right\} - p_{iT-1}q_{iT-1} + \gamma E_{iT-1} [p^*_iT] q_{iT-1}.
$$

The last term shows up because the additional tree today would save the purchase tomorrow if and only if the game continues. Hence, the optimal order at $T - 1$ is

$$
q_{iT-1} (p_{iT-1}; F_{iT-1}) = \frac{(1 - \gamma) E_{iT-1} [v] + \gamma E_{iT-1} [p^*_iT] - p_{iT-1}}{\rho (1 - \gamma)} - x_{iT-2}.
$$

From (27),

$$
E_{iT-1} [p^*_iT] = \sum_{k=0}^{T-1} \frac{\beta_{sTk}}{\beta_{pT}} E_{iT-1} [s_{k,i}] \\
= \frac{\sqrt{u}\lambda T}{\tau e + \chi T} E_{iT-1} [v_A].
$$
Hence,

\[ q_{iT-1}(p_{iT-1}; F_{iT-1}) = \frac{1}{\rho(1-\gamma)} \left\{ \left( 1 - \gamma + \frac{\chi_T}{\tau_e} + \chi_T \right) \sqrt{u} E_{iT-1} [v_A] - p_{iT-1} \right\} - x_{iT-2} \]

\[ = \frac{1}{\rho} \left\{ \left( 1 + \frac{\gamma}{1-\gamma} \frac{\chi_T}{\tau_e} \right) \sqrt{u} E_{iT-1} [v_A] - \frac{p_{iT-1}}{1-\gamma} \right\} - x_{iT-2}. \]

Finally, recall from (26) that

\[ E_{iT-1} [v_A] = \frac{1}{\tau_e + \chi_T} \left( \sum_{k=0}^{T-2} n^k \varphi_k s_{ik} + n^{T-1} \varphi_{T-1, k} \beta_{pT-1} p_{iT-1} \right). \]

Therefore,

\[ q_{iT-1}(p_{iT-1}; F_{iT-1}) = \left( 1 + \frac{\gamma}{1-\gamma} \frac{\chi_T}{\tau_e} \right) \frac{\sqrt{u}}{\rho} \sum_{k=0}^{T-2} n^k \varphi_k s_{ik} \left( \sqrt{u} n^{T-1} \varphi_{T-1, k} \beta_{pT-1} \right) \frac{p_{iT-1}}{\tau_e + \chi_T} - x_{iT-2} \]

\[ = \frac{1}{\rho} \left\{ \left( 1 + \frac{\gamma}{1-\gamma} \frac{\chi_T}{\tau_e} \right) \sqrt{u} n^{T-1} \varphi_{T-1, k} \beta_{pT-1} \frac{p_{iT-1}}{\tau_e + \chi_T} \right\} - x_{iT-2}. \]

By equating coefficients with those in (21) for \( t = T - 1 \),

\[ \beta_{sT-1} = \left( 1 + \frac{\gamma}{1-\gamma} \frac{\chi_T}{\tau_e} \right) \frac{\sqrt{u}}{\rho} n^k \varphi_k, \quad k = 0, \ldots, T - 2, \]

\[ \beta_{sT-1} = 1, \]

\[ \beta_{pT-1} = \frac{1}{\rho} \left\{ \left( 1 + \frac{\gamma}{1-\gamma} \frac{\chi_T}{\tau_e} \right) \sqrt{u} n^{T-1} \varphi_{T-1} \beta_{pT-1} \frac{p_{iT-1}}{T-2} \sum_{k=0}^{T-2} \beta_{sT-1} \right\}. \]
From the expression of $\beta_{sT-1k}$,

$$
\sum_{k=0}^{T-2} \beta_{sT-1k} = \left( 1 + \frac{\gamma}{1 - \gamma \frac{\tau_u}{\tau_x} + \chi_T} \right) \frac{\sqrt{u}}{\rho} \chi_{T-2}^{1 - 1, \chi_T},
$$

$$
\tilde{\beta}_{sT-1k} = \frac{n^k \varphi_k}{\chi_{T-2}}, \quad k = 0, \ldots, T-2,
$$

$$
\tilde{\beta}_{xT-1} = \left( 1 + \frac{\gamma}{1 - \gamma \frac{\tau_u}{\tau_x} + \chi_T} \right)^{-1} \frac{\sqrt{u}}{\rho} \chi_{T-2},
$$

$$
\beta_{pT-1} = \frac{1}{\rho} \left\{ \frac{1}{1 - \gamma} - \sqrt{\frac{u n^{T-1} \varphi_{T-1}}{1 - \gamma + \chi_{T-1}}} \left( \frac{\sqrt{u} \chi_{T-2}}{\rho} \frac{\tau_u}{\tau_x} + \chi_{T-1} \right)^{-1} \beta_{pT-1} \right\}
$$

$$
= \frac{1}{\rho} \left\{ \frac{1}{1 - \gamma} - \rho \frac{\chi_{T-2} \varphi_{T-1}}{\chi_{T-2}} \beta_{pT-1} \right\}.
$$

The last condition can be solved for $\beta_{pT-1}$:

$$
\beta_{pT-1} = \frac{1}{(1 - \gamma)} \rho \left( 1 + \frac{n^{T-1} \varphi_{T-1}}{\chi_{T-2}} \right)^{-1}
$$

$$
= \frac{1}{(1 - \gamma)} \frac{\chi_{T-2}}{\rho}.
$$

We define $\Gamma_1 \equiv 1 + \frac{\gamma}{1 - \gamma \frac{\tau_u}{\tau_x} + \chi_T}$ to write

$$
q_{iT-1}(p_{iT-1}; F_{iT-1}) = \frac{1}{\rho} \sqrt{u} \frac{T-2}{k=0} \sum_{k=0}^{T-2} n^k \varphi_k s_{ik} - x_{iT-2} - \frac{1}{(1 - \gamma)} \frac{\chi_{T-2}}{\rho} p_{iT-1}.
$$

Note that $\Gamma_1 \in \left( 1, \frac{1}{1 - \gamma} \right)$.

Proceeding similarly as before,

$$
\frac{1}{n^{T-2} \varphi_{T-1}} = \frac{\tau_{T-2}}{\rho} \frac{\tau_{xT-2}}{\rho} \left( \frac{\chi_{T-2}}{\rho} \frac{\tau_u}{\tau_x} + \chi_{T-1} \right)^2 \frac{\tau_{\xi}}{\tau_{xT-2}}
$$

$$
= \frac{1}{\chi_{T-2}} + \rho \frac{\tau_{\xi}}{u} \frac{\tau_{xT-2}}{\rho} \left( \frac{\chi_{T-2}}{\rho} \frac{\tau_u}{\tau_x} + \chi_{T-1} \right)^2.
$$
This can be seen as an equation in \( \varphi_{T-1} \) given \((\varphi_1, \ldots, \varphi_{T-2}, \varphi_T)\) and \(\tau_{xT-2}\):

\[
\frac{n^{T-2}\varphi_{T-1}}{\chi_{T-2}} + 1 + \frac{\rho^2}{u} \frac{\tau_{\varepsilon}}{\tau_{xT-2}} \left( \frac{1}{\Gamma_1} \right) \frac{\tau_{\varepsilon} + \chi_{T-2}}{\chi_{T-2}} n^{T-2} \varphi_{T-1} = 0.
\]

\(\Leftrightarrow \frac{\rho^2}{u} \frac{\tau_{\varepsilon}}{\tau_{xT-2}} \left( \frac{1}{\Gamma_1} \right) \frac{\tau_{\varepsilon} + \chi_{T-2}}{\chi_{T-2}} n^{T-2} \varphi_{T-1} = 1 - \frac{n^{T-2}}{\chi_{T-2}} \varphi_{T-1}.
\]

Because the right hand side must be positive for the solution to exist,

\[
\frac{n^{T-2}\varphi_{T-1}}{1 - \frac{n^{T-2}}{\chi_{T-2}} \varphi_{T-1}} = \frac{u}{\rho^2} \frac{\tau_{xT-2}}{\tau_{\varepsilon}} \left( \frac{1}{\Gamma_1} \frac{\chi_{T-2}}{\chi_{T-2}} \right)^2.
\]

(31)

Given \((\varphi_1, \ldots, \varphi_{T-2})\), the left hand side of (31) is increasing in \(\varphi_{T-1}\) and continuously change from zero to positive infinity for \(\varphi_{T-1} \in \left[0, \frac{\chi_{T-2}}{n^{T-2}}\right]\). On the other hand, the right hand side is decreasing in \(\varphi_{T-1}\). Hence, there is a unique \(\varphi_{T-1}^*\) that solves (31) for any given \((\varphi_1, \ldots, \varphi_{T-2}, \varphi_T)\) and \(\tau_{xT-2}\).

Also, substituting derived coefficients into (25) gives

\[
\frac{\tau_{\varepsilon}}{\tau_{xT-1}} = \frac{\sqrt{u}}{\rho} \frac{\sqrt{\chi_{T-2}}}{\chi_{T-1}} \left( \frac{1}{n^{T-1} \varphi_{T-1}} + \sum_{k=0}^{T-2} \left( \frac{n^k \varphi_k}{\chi_{T-2}} \right)^2 \right)
\]

\[
= \frac{u}{\rho^2} \frac{\sqrt{\chi_{T-2}}}{\chi_{T-1}} \left( \frac{1}{n^{T-1} \varphi_{T-1}} + \frac{1}{\chi_{T-2}} \right)
\]

\[
= \frac{u}{\rho^2} \frac{\sqrt{\chi_{T-2}}}{\chi_{T-1}} \left( \frac{1}{n^{T-1} \varphi_{T-1}} \right) \left( \frac{\chi_{T-1}}{n^{T-1} \varphi_{T-1} \chi_{T-2}} \right)
\]

\[
= \frac{u}{\rho^2} \frac{\chi_{T-2} \chi_{T-1}}{n^{T-1} \varphi_{T-1}} \left( \frac{\Gamma_1}{\sqrt{\chi_{T-2}} + \chi_{T-1}} \right)^2.
\]

\(\Gamma_1\) as a function of \(\varphi_1, \ldots, \varphi_T\) is increasing in each argument if \(\gamma > 0\), but \(\frac{\sqrt{\chi_{T-2}}}{\sqrt{\chi_{T-2}} + \chi_{T-1}}\) is decreasing in \(\varphi_{T-1}\).
This last expression can be substituted into (30) to obtain

$$\frac{n^{T-1} \varphi_T}{1 - \frac{n^{T-1}}{\lambda T_{T-1}} \varphi_T} = \frac{n^{T-1} \varphi_{T-1}}{\lambda T_{T-2} \lambda T_{T-1}} \frac{\left( \frac{T_k}{\tau_e} + \lambda T_{T-1} \right)^2}{\left( \frac{T_k}{\tau_e} + \lambda T_{T-1} \right)^2}.$$  

$$\Leftrightarrow \varphi_T = \frac{1}{\lambda T_{T-2}} \left( \lambda T_{T-1} - n^{T-1} \varphi_T \right) \left( \frac{1}{\lambda T_{T-2}} \left( \frac{T_k}{\tau_e} + \lambda T_{T-1} \right)^2 \varphi_{T-1} \right).$$  

$$\Leftrightarrow \varphi_T = \left( 1 + \frac{n^{T-1}}{\lambda T_{T-2}} \left( \varphi_{T-1} - \varphi_T \right) \right) \left( \frac{1}{\lambda T_{T-2}} \left( \frac{T_k}{\tau_e} + \lambda T_{T-1} \right)^2 \varphi_{T-1} \right).$$  

Because $\frac{1}{\lambda T_{T-2}} \left( \frac{T_k}{\tau_e} + \lambda T_{T-1} \right)^2 < 1$, this shows that $\varphi_T < \varphi_{T-1}$ in equilibrium.

The rest of the characterization of the dynamic equilibrium uses an induction argument.

We define a sequence $\{\Gamma_{T-t}\}_{t=1}^{T} = \{\Gamma_{T-1}, \Gamma_{T-2}, \ldots, \Gamma_1, \Gamma_0\}$ as follows. First, $\Gamma_0 \equiv 1$. For $t = 1, \ldots, T - 1,$

$$\Gamma_{T-t} \equiv 1 + \frac{\gamma}{1 - \gamma} \left\{ t = T - 1 \right\} \frac{\lambda T_{t+1}}{\lambda T_{T-(t+1)}} + \lambda T_{T-(t+1)}.$$  

\[\text{(32)}\]

**Proposition** For $t = 1, \ldots, T,$

(a) $\beta_{st_k} = \frac{\sqrt{T}}{\rho} \frac{\Gamma_{T-t}}{\lambda T_{T-t}} n^k \varphi_k$ for $k = 1, \ldots, t - 1$, $\beta_{st} = 1$, and $\beta_{pt} = \frac{1}{\rho (1 - \gamma) (1 < T)} \frac{\lambda T_{t-1}}{\lambda T_{t}}$.

(b) For $t = 1$, $\varphi_1$ solves

$$\frac{\varphi_1}{1 - \varphi_1} = \frac{u T_x}{\rho^2 \tau_e} \left( \frac{\Gamma_{T-1}}{\lambda T_{T-2}} \right)^2.$$  

\[\text{(33)}\]

For $t = 2, \ldots, T$, $\varphi_t$ solves

$$\varphi_t = \left( 1 + \frac{n^{t-1}}{\lambda T_{T-2}} \left( \varphi_{t-1} - \varphi_t \right) \right) \left( \frac{\Gamma_{T-t}}{\lambda T_{T-(t-1)}} \left( \frac{T_k}{\tau_e} + \lambda T_{T-(t-1)} \right)^2 \varphi_{t-1} \right).$$  

\[\text{(34)}\]

(c) The distribution of the tree at the end of time $t$ is $N \left( 0, \tau_{xt}^{-1} \right)$, where

$$\frac{\tau_e}{\tau_{xt}} = \frac{u \lambda T_{t-1}}{\rho^2 n^t \varphi_t} \left( \frac{\Gamma_{T-t}}{\lambda T_t} \right)^2.$$  

31
Note that \( \{\Gamma_{T-t}\}_{t=1}^{T-1} \) may depend on \( \{\varphi_t\}_{t=1}^T \), but when they do they are continuous in \( \{\varphi_t\}_{t=1}^T \) and bounded in \( \left(1, \frac{1}{1-\gamma}\right) \). Therefore, (33) and (34) jointly define a continuous mapping from \( \mathbb{R}^T \) into itself, whose fixed point characterizes the equilibrium value of \( \{\varphi_t\}_{t=1}^T \). For any \( T \), a fixed point exists since \( \varphi_t \in [0, \overline{\varphi}] \) is bounded.\(^{10}\) However, it must be solved numerically.

**UNDER CONSTRUCTION.**

We provide a series of lemmas that characterize the dynamic equilibrium.

**Lemma 5 (\( \Gamma_{T-t} \))**

(a) For \( t = 1, \ldots, T - 1 \), \( \Gamma_{T-t} \in \left[ \frac{\tau_t + \chi_{t+1}}{\tau_t^2 + (1-\gamma)\chi_{t+1}}, \frac{(1-\gamma)\tau_t + \chi_{t+1}}{(1-\gamma)(\tau_t^2 + \chi_{t+1})} \right] \).

If \( \frac{\tau_t}{\tau_e} = 0 \), \( \Gamma_{T-t} = \frac{1}{1-\gamma} \forall t = 1, \ldots, T - 1 \).

If \( \gamma = 0 \), \( \Gamma_{T-t} = 1 \forall t = 1, \ldots, T \).

(b) Given \( \gamma \frac{\tau_t}{\tau_e} > 0 \), \( \frac{\Gamma_{T-t}}{\Gamma_{T-(t-1)}} \in \left(1, \frac{\tau_t + \chi_t}{(1-\gamma)\tau_e^2 + \chi_t} \right) \) for \( t = 1, \ldots, T - 1 \).

(c) As \( \gamma \to 1 \), \( \Gamma_{T-t} \to \infty \) for \( t = 1, \ldots, T - 1 \).

Next, define \( K_t \equiv \frac{\Gamma_{T-t}}{\Gamma_{T-(t-1)}} \frac{\tau_t + \chi_{t-1}}{\tau_e^2 + \chi_t} \).

**Lemma 6 (learning speed \( \frac{\varphi_t}{\varphi_{t-1}} \))**

(a) \( \frac{\varphi_t}{\varphi_{t-1}} < 1 \).

For each \( t = 2, \ldots, T - 1 \), only one of the following three cases is possible:

\[
\begin{align*}
(i) \quad 1 &< \frac{\varphi_t}{\varphi_{t-1}} < K_t^2, \\
(ii) \quad 1 &> \frac{\varphi_t}{\varphi_{t-1}} > K_t^2, \\
(iii) \quad 1 &= \frac{\varphi_t}{\varphi_{t-1}} = K_t.
\end{align*}
\]

\(^{10}\)The upper bound \( \overline{\varphi} \) is provided by the following recursive relationship: \( \overline{\varphi}_{t+1} = \frac{1}{\eta_t} \sum_{k=0}^{t} n^k \overline{\varphi}_k \) with \( \overline{\varphi}_1 = 1 \).
If $\gamma_{\tau} = 0$, case (ii) holds $\forall t = 2, ..., T - 1$. Given $\frac{\gamma_{\tau}}{\tau} > 0$, as $\gamma \to 1$, case (i) holds $\forall t = 2, ..., T - 1$.

(b) $K_t = \frac{\gamma_{\tau} + \chi_{t-1}}{\tau_{\gamma} + (1 + \gamma_{T-t})\chi_t} \Gamma_{T-t}$.

Lemma 7 (allocation convergence $\frac{\text{Var}[x_t]}{\text{Var}[x_{t-1}]}$)

$\forall t = 2, ..., T$,

(a) $\frac{\text{Var}[x_t]}{\text{Var}[x_{t-1}]} = \frac{\chi_{t-1} + n^t \varphi_t}{n \chi_{t-1} - n^t \varphi_t} > \frac{1}{n}$.

(b) $\frac{\text{Var}[x_t]}{\text{Var}[x_{t-1}]} \leq 1 \iff \varphi_t \in \left(0, \frac{n-1}{2n} \frac{\chi_{t-1}}{n^t-1}\right)$ and $\frac{\text{Var}[x_t]}{\text{Var}[x_{t-1}]} > 1 \iff \varphi_t \in \left(\frac{n-1}{2n} \frac{\chi_{t-1}}{n^t-1}, \frac{\chi_{t-1}}{n^t}\right)$.

In particular, $\frac{\text{Var}[x_t]}{\text{Var}[x_{t-1}]} > 1$ if $n = 1$.

Lemma 7(a) shows that the change in the dispersion of the allocation is bounded below by $\frac{1}{n}$ and decreasing in $\varphi_t$. The lower bound $\frac{1}{n}$ is attained by the case with symmetric information or $\varphi_t \to 0$. Lemma 7(b) shows that the dispersion can increase, and it happens if and only if there is sufficient amount of information aggregation in that period.

5.3 Simulations

UNDER CONSTRUCTION.

We numerically solve the model for $T = 10$, $n \in \{1, 2, 3\}$, and $\gamma \in \{0, 0.125, 0.25, 0.5\}$. The solutions show that $\tau_t$ increases in $\gamma$ while $\tau_{xt}$ decreases in $\gamma$ for given $n$ and $t$. Thus, as traders become more concerned with future trading rounds, more information is aggregated, while allocation becomes more dispersed. Importantly, the allocation dynamics can be quite non-monotonic as in Figure 1. In some cases, the dispersion can be lower than the initial dispersion at some point in the trading rounds. If the game ends at such time, traders learn about the tree from trading, but end up with the more dispersed tree allocation and the higher inventory cost.
6 Conclusion

This paper studied dynamic asset trading with two frictions: 1) trading is locally intermediated, and 2) all traders have private information. We analyze how dispersed information is aggregated over time from intermediated trades. In the absence of private signals, the ex post efficient allocation can be achieved after the sufficient number of trading rounds given that more than two traders are involved in each intermediated trading. With private signals, information gets aggregated over time, but the allocation may diverge away from the efficient allocation.

7 Appendix 1. Under construction.

We solve the model for $T = 10$ and $u = 1$. Other model parameters are set as follows.

$$
\tau_\varepsilon = 0.05 \quad \tau_x = 5 \quad n \in \{1, 2, 3\}
$$

$$
\tau_v = 1 \quad \rho = 2.5 \quad \gamma \in \{0, 0.125, 0.25, 0.5\}
$$

8 Appendix 2. Under construction.

Proof of Lemma 1.

(b) $E \left[ |q_i^{xp}| \right] = \frac{n}{n+1} E \left[ \frac{1}{n} \sum_{j \neq i} x_{j0} - x_{i0} \right] = \frac{n}{n+1} \sqrt{\frac{2}{\pi} V \left[ \frac{1}{n} \sum_{j \neq i} x_{j0} - x_{i0} \right]} = \sqrt{\frac{2}{\pi} \frac{n}{n+1} \frac{1}{\tau_x}}.$
References


