Auctioning risk: The all-pay auction under mean-variance preferences*

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Abstract

We analyse the all-pay auction with incomplete information and variance-averse bidders. We characterise the symmetric equilibrium for general distributions of valuations and any number of bidders. Variance aversion is a sufficient assumption to predict that high-valuation bidders increase their bids relative to the risk-neutral case while low types decrease their bid. Considering an asymmetric two-player environment with uniform valuations, we show that a more variance-averse type always bids higher than her less variance-averse counterpart. Taking mean-variance bidding behaviour as given, we show that an expected revenue maximising seller may want to optimally limit the number of participants. (JEL *C7, D7, D81.* Keywords: *Auctions, Contests, Mean-Variance preferences.*)

1 Introduction

Mean-variance preferences (Markowitz, 1952) have long been successfully applied to portfolio choice investment problems where asset managers evaluate alternative portfolios on the basis of the mean and variance of their return. It therefore may be surprising that the mechanism design literature and, specifically, the large literature on auctions has not yet addressed the decision making problem of players endowed with mean-variance preferences over their wealth. The present paper attempts to close this gap by fully characterising bidding and revenue-optimal sales behaviour in one of the standard auction types, the all-pay auction. This auction type may be viewed as a natural candidate because it exposes a bidder to the inherent risk of either winning the object (potentially at a bargain) or losing one's bid without gaining anything.

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We offer a group of motivational examples underpinning the relevance of this approach. All of our examples have two random components: i) the 'endogenous' variance centred on the bidding process, i.e., the tension between winning a prize (consisting of a private valuation minus the own bid) and paying the own bid for sure, and ii) the 'exogenous' variance of the prize (or outside option) itself. Examples for both seem to abound in, for instance, a firm's market development decisions while facing competitors involved in similar decision problems. A further and distinct set of examples lies in the practice of introductory offers ('burning money') with which firms try to ascertain uncertain future market shares through certain upfront losses. Our setup is as applicable to an entrepreneur's choice of team composition as it is to the research portfolio selection of heads of R&D or similar institutions. Similarly, our model seems to fit well with patent races under rivalry as analysed, for instance, by Dasgupta (1986) in a full information setup. Finally, the classical portfolio selection problem seems to be related as, clearly, portfolio choice is usually based on mean-variance considerations and anecdotal evidence is available which underlines all-pay aspects of fund managing practice.[1]

What is a general motivation to consider risk aversion in winner-pay auctions? A bidder in a (first-price) winner-pay auction controls, through her bid, both the probability of winning and the amount she wins. A risk averse bidder is willing to sacrifice some of this payoff (the individual value minus her bid) for a higher probability of winning (through a higher bid). Hence, in a first-price, winner-pay auction, risk aversion causes an increase in equilibrium bids relative to the risk neutral case.[2] In all-pay auctions, in addition, increasing one's bid has the direct negative effect of increasing the certain payment *independently* of both other effects. In consequence, a low valuation bidder under risk aversion wants to *decrease* her losses while a risk averse, high-type bidder wants to *increase* her probability of winning through more aggressive bidding in the all-pay auction.

Apart from intellectual curiosity, we field three main arguments in order to justify the attention we place on mean-variance preferences in this paper. First, the typically employed risk-neutral, expected payoff analysis of auctions simply ignores any risk considerations; compared with that, a mean-variance analysis certainly represents progress. Second, if all relevant probability distributions have the same elliptic form, then the mean and variance represent a sufficient statistic to identify the true distribution of returns. Then, the mean-variance approach does not differ from a *full* account of expected utility using a general representation of risk aversion. Third, financial practitioners make the vast majority of their day-to-day portfolio choice decisions on the basis of the mean and variance of portfolios. It would seem likely that this group could benefit from a similar representation of their choices for auctioning activities.

[1] "Five-star funds. Four-star funds. Those seem to be the only mutual funds people want to buy." Investors Business Daily, "Making Money in Mutuals: Don't Focus too Narrowly on Star Ratings," by A. Shell, 22 June 1998, cited in Bagnoli and Watts (2000). Hence, although all funds invest efforts, only the most highly ranked funds obtain large investments.

[2] See Maskin and Riley (1984).

Literature

To the best of the authors' knowledge there are no existing papers which analyse auctions or incomplete information contests under mean-variance preferences. Most existing work on risk aversion in contests applies to full information Tullock contests.[3] An attempt to model mean-variance preferences in this full information case is Robson (2012) who derives an 'irrelevance result' in the sense that for two-player Tullock contests bidding behaviour is not affected by the introduction of an aversion to variance. A more general analysis in terms of risk aversion of the same setup is Cornes and Hartley (2012b) who focus on existence questions of both symmetric and asymmetric Nash equilibria. (For the case of loss aversion see Cornes and Hartley (2012a).) The only existing works on risk aversion for the incomplete information all-pay auction of which we are aware are Fibich, Gavious, and Sela (2006), Cingottini and Menicucci (2006), and Parreiras and Rubinchik (2010).[4] Fibich, Gavious, and Sela (2006) show that an analytic characterisation of equilibrium strategies cannot be usually obtained for von Neumann-Morgenstern risk-averse players. Thus, contrasting our fully analytical approach, they turn to perturbation analysis to obtain their mostly numerical results. Esö and White (2004) show that under special conditions on valuations, decreasingly absolute risk averse players prefer the first-price auction to the all-pay auction. Fibich, Gavious, and Sela (2006) extend this ranking to the case of general risk aversion for independent valuations. Their results are limited, however, by the fact that they cannot generally obtain analytic forms of the equilibrium bidding strategies of risk averse players. We can overcome this limitation at the price of focusing attention to the class of linear mean-variance preferences.

Cingottini and Menicucci (2006) study an environment composed of ex-ante symmetric bidders sharing the same preferences exhibiting constant absolute risk aversion. They find that it is revenue-optimal for the seller to exclude all but two randomly chosen competitors. Their result, which is contrary to the monotonicity of revenue in the risk neutral case, is obtained provided that bidders are either highly risk averse or very likely to possess a particular, known valuation.

Parreiras and Rubinchik (2010) analyse bidding behaviour in contests where three or more players draw their valuations from asymmetric supports and may have asymmetric attitudes toward risk. They find that these ex-ante asymmetries may lead to player drop-out or, for sufficiently risk-averse players, the use of discontinuous 'all-or-nothing' strategies. Thus, both cases exhibit behaviour which is very different from the standard ex-ante symmetric equilibrium case. Although the authors cannot explicitly determine the equilibrium bidding functions in general, they construct a simple check for wether or not a particular bid can be part of a player's equilibrium strategy. This test is used to establish the above participation conclusions.

Papers relating to the analysis of risk aversion in general winner-pay auction environments are Maskin and Riley (1984) and Matthews (1987), both discussing risk-averse bidders' behaviour in auctions, Esö and White (2004), analysing precautionary bidding in auctions, and Esö and Futó (1999), who derive the revenue-optimal strategy for a risk-averse seller, and Hu, Matthews, and

[3] An analysis of the (repeated) full information case for more general success probabilities is Ireland (2004).

[4] It may be worth pointing out that the analysis of risk-aversion in Lazear and Rosen (1981) also boils down to mean-variance preference analysis as their output noise term is fully characterised by the mean and variance of the Normal distribution.

Zou (2010) who discuss reserve prices. The existing analyses of asymmetric auctions, for instance Amann and Leininger (1996), Lizzeri and Persico (2000), Maskin and Riley (2000), Fibich, Gavious, and Sela (2004), Parreiras and Rubinchik (2010), Kirkegaard (2012), or Kaplan and Zamir (2012), typically employ asymmetric distributions (or supports) while we use our idiosyncratic variance-aversion parameter. Next to Parreiras and Rubinchik (2010), to the best of the authors' knowledge, this is the only paper to analyse bidding in a contest when players are asymmetric in their degree of risk-aversion. For accounts of auctions under ambiguity see, for instance, Bose and Daripa (2009), and for a more general approach to mechanism design under ambiguity, see Bodoh-Creed (2012).

In terms of revenue and payoff analysis, Matthews (1987) compares payoffs for risk averse behaviour when bidders exhibit constant and increasing absolute risk aversion (CARA and IARA, respectively). For CARA, he finds that bidders are indifferent between first- and second-price auctions, while for IARA bidders prefer the first-price auction. As discussed, Cingottini and Menicucci (2006) find that revenue is maximal for ex-ante symmetric bidders exhibiting CARA preferences. Smith and Levin (1996) show that this ranking can be reversed under decreasing absolute risk aversion.

The present paper is dealing with variance aversion which, in general, is different from risk aversion.[5] Mean-variance preferences can be transformed into expected utility form under certain assumptions on the location, scale, and concordance parameters of the environment. For the precise relation of von Neumann-Morgenstern preferences to mean-variance preferences, see, for instance, Sinn (1983), Kroll, Levy, and Markowitz (1984), Mayer (1987), or, more recently, Eichner (2008), or Eichner and Wagener (2009).

2 The model

There is a seller with one indivisible object for sale. The seller's valuation of the item is (normalised to) zero. There are $n \geq 2$ potential buyers with valuations θ_i, $i \in \mathcal{N} = \{1, 2, \ldots, n\}$, respectively. The own valuation is private information of each buyer and all players' valuations, θ_i, $i \in \mathcal{N}$, are assumed to be independent draws from the same increasing and atom-less distribution F. Let $f(\cdot) = F'(\cdot)$ represent the associated probability density function and $[\underline{\theta}, \bar{\theta}] = [0, 1]$ its support. In section 3.2 we consider the possibility that the final value of the object may further be influenced by an exogenous shock, $\varepsilon \sim W(0, \hat{\varepsilon}^2)$, which is distributed over some compact interval with mean zero and variance $\hat{\varepsilon}^2$. Similarly, a player's valuation of the state in which she does not win the object may be subject to another independently distributed exogenous shock $\delta \sim L(0, \hat{\delta}^2)$.[6]

After realising their own (expected) valuations of the object, θ_i, all players simultaneously submit their bids, b_i, $i \in \mathcal{N}$. The player with the highest bid receives the object and all players forgo their

[5] For a discussion of the differences see, for instance, Rothschild and Stiglitz (1970). Under the specific modelling assumptions made below, however, the two can be equated and, as we hope, important insights into the behaviour under general risk aversion can be deduced from our simple model even when these assumptions are not met.

[6] As an example of an interaction in which a discrimination between shocks is natural consider patent races where firms are already in the market. Here, the success of one firm affects the market prospects of the losers.

bids. After the auction has ended, the exogenous shocks realise and player i's payoff is given by

$$\pi_i(b_i, b_{-i}; \theta_i) = \begin{cases} \theta_i + \varepsilon - b_i & \text{if } b_i > b_j \forall j \neq i, \\ \frac{1}{m}(\theta_i + \varepsilon) + \frac{m-1}{m}\delta - b_i & \text{if } i \in Q = \{j \in \mathcal{N} | b_j = \max_{k \in \mathcal{N}} b_k\}, m = |Q|, \\ \delta - b_i & \text{if } \exists j : b_i < b_j. \end{cases}$$

In the following we focus on three particular cases.

1. *No exogenous shock*: In this case, both ε and δ take the value zero with probability one, i.e., each bidder $i \in \mathcal{N}$, knows with certainty that in case of winning the auction she will obtain a prize of value θ_i while her valuation of losing is zero.
2. *Winner's uncertainty*: When $\hat{\varepsilon}^2 > 0$, the valuation of the prize is uncertain and θ_i is merely a signal, the expected value of the object.
3. *Loser's uncertainty*: In case that $\hat{\delta}^2 > 0$, a player faces uncertainty in the event that she does not secure the object for sale.

Notice that the three cases described above do not have any effects on equilibrium bidding behaviour in the standard model of buyers with risk-neutral von Neumann-Morgenstern utility, who simply maximise expected payoffs. In the following we discuss how bidding strategies of buyers with mean-variance preferences are affected in each of the aforementioned scenarios. For much of our analysis we focus solely on the effects of the endogenous variance which is present in any all-pay auction. In section 3.2 we discuss how the addition of exogenous shocks (cases 1 and 2) further alters bidding strategies of variance-averse players.

When buyers have mean-variance preferences, they maximise an objective function $u_i(\mu_i, \sigma_i^2)$, which is increasing in the expected payoff, μ_i, and decreasing in the variance of their payoff, σ_i^2. For our analytical investigation we use the following simple linear representation of mean-variance preferences[7]

$$u_i(\mu(b, \theta_i), \sigma^2(b, \theta_i)) = \mu(b, \theta_i) - \nu_i \sigma^2(b, \theta_i), \tag{1}$$

where the parameter $\nu_i \in [0, 1]$ accounts for player i's variance-aversion. The case of $\nu_i = 0$ represents the standard case of risk-neutral expected payoff maximisation. Bounding the degree of variance aversion from above guarantees the existence of a pure strategy equilibrium.

Provided a player's knowledge of his own type, his bid can be interpreted as choosing a lottery with two possible outcomes. If the player submits the highest bid, he will in expectation receive his valuation of the prize (represented by his type) minus the cost of his bid. In all other events, he will loose his bid. Note that the payoff difference between these two outcomes remains fixed for any bid and is just equal to the player's type. However, selecting a higher bid does not only decrease the respective payoffs for both outcomes, but also moves probability mass from the loosing to the winning outcome.

[7] A large body of empirical and theoretical work employs variants of this simple form on the basis of both tractability and testability. For discussions see, for instance, Tsiang (1972), Coyle (1992), Saha (1997) or the textbook treatment in Sargent and Heller (1987, p154–5). Recently, Chiu (2010) discusses the applicability of mean-variance preferences of this form to a large set of problems in finance and economics in choice theoretic terms.

3 Bidding behaviour

3.1 The symmetric case: n identical bidders

Under the first-price, all-pay auction, a type-θ_i bidder's expected payoff when issuing a bid of b is given by

$$\pi(b,\beta;\theta_i)=\int_0^{\beta^{-1}(b)}\cdots\int_0^{\beta^{-1}(b)}\theta_i f(\theta_1)\cdots f(\theta_{n-1})d\theta_1\cdots d\theta_{n-1}-b \tag{2}$$

where $\beta(\theta)$ is the tentative symmetric equilibrium bid issued by a type-θ player. We conjecture that the function $\beta(\theta)$ is non-decreasing and denote the highest type who submits a bid no higher than b by $\theta=\beta^{-1}(b)$. It is well known, for instance from Milgrom (2004, p119), that the strategies

$$\beta_{\mathsf{rn}}(\theta)=\theta(F(\theta))^{n-1}-\int_0^{\theta}(F(\vartheta))^{n-1}d\vartheta, \tag{3}$$

maximise (2), hence constituting a symmetric equilibrium if players simply maximise their expected payoffs (i.e., $\nu_i=0$ for all $i\in\mathcal{N}$).

With mean-variance preferences, symmetric players with $\nu\equiv\nu_1=\cdots=\nu_n$ choose a bidding function which maximises (1), taking into account their payoff variance in addition to their expected payoff. These are given for the first-price, all-pay auction as

$$\begin{aligned}\mu(b,\theta_i)&=\theta_i(F(\beta^{-1}(b)))^{n-1}-b;\\ \sigma^2(b,\theta_i)&=(F(\beta^{-1}(b)))^{n-1}(\theta_i-b-\mu)^2+(1-(F(\beta^{-1}(b)))^{n-1})(-b-\mu)^2\\ &=(F(\beta^{-1}(b)))^{n-1}(1-(F(\beta^{-1}(b)))^{n-1})\theta_i^2.\end{aligned} \tag{4}$$

Inserting these back into the player's objective[8] and rearranging gives

$$u_i(b,\theta_i)=\theta_i(F(\beta^{-1}(b)))^{n-1}(1-\nu\theta_i+\nu\theta_i(F(\beta^{-1}(b)))^{n-1})-b. \tag{5}$$

The first-order condition for maximisation of (5) with respect to b is[9]

$$\theta_i\left(1-\nu\theta_i+2\nu\theta_i(F(\beta^{-1}(b)))^{n-1}\right)(n-1)(F(\beta^{-1}(b)))^{n-2}f(\beta^{-1}(b))\frac{\partial\beta^{-1}(b)}{\partial b}=1. \tag{8}$$

In the symmetric equilibrium $b=\beta(\theta_i)$, this yields the first-order differential equation

$$\beta'(\theta_i)=\theta_i\left(1-\nu\theta_i+2\nu\theta_i(F(\theta_i))^{n-1}\right)(n-1)(F(\theta_i))^{n-2}f(\theta_i). \tag{9}$$

[8] Note that in our model using the modified mean-variance approach due to Blavatskyy (2010) would lead to qualitatively the same results since the mean absolute semideviation is $r(b,\theta_i)=(F(\beta^{-1}(b)))^{n-1}(1-(F(\beta^{-1}(b)))^{n-1})\theta_i$.

[9] The second-order condition for the case of two players is

$$\frac{2\theta_i\nu-4\theta_i\nu F(\theta_i)-1}{\theta_i^2 f(\theta_i)(1-\theta_i\nu+2\theta_i\nu F(\theta_i))^2}<0 \tag{6}$$

The condition for the general case of $n>2$ players is more involved and relegated to the appendix. In our model $\theta_i\nu\leq 1$, therefore, a sufficient condition for the general second-order condition to hold is that

$$F(\theta)^{n-1}>\frac{1}{2}-\frac{1}{4\theta\nu}. \tag{7}$$

This condition (7) generally holds if ν is sufficiently small such that $\theta\nu\leq\frac{1}{2}$ for all θ in the support of the type distribution $F(\cdot)$, but otherwise imposes a restriction on the distribution of types and/or the number of players.

This differential equation together with the boundary condition $\beta(0) = 0$ is solved (through repeated integration by parts) by the bidding function

$$\begin{aligned}\beta_{\mathsf{mv}}(\theta_i) =& \theta_i (F(\theta_i))^{n-1} - \int_0^{\theta_i} (F(\vartheta))^{n-1}\, d\vartheta - \nu\theta_i^2 (F(\theta_i))^{n-1} + \\ & \nu\theta_i^2 (F(\theta_i))^{2(n-1)} + \nu \int_0^{\theta_i} 2\vartheta (F(\theta))^{n-1}\, d\vartheta - \nu \int_0^{\theta_i} 2\vartheta (F(\theta))^{2(n-1)}\, d\vartheta \\ =& \beta_{\mathsf{rn}} - \nu \left(\theta_i^2 \left(F(\theta_i)^{n-1} - (F(\theta_i))^{2(n-1)} \right) - \int_0^{\theta_i} 2\vartheta \left((F(\vartheta))^{n-1} - (F(\vartheta))^{2(n-1)} \right) d\vartheta \right) \\ =& \beta_{\mathsf{rn}} - \nu \int_0^{\theta_i} \vartheta^2 (n-1)(F(\vartheta))^{n-2} f(\vartheta) \left(1 - 2(F(\vartheta))^{n-1}\right) d\vartheta.\end{aligned} \tag{10}$$

Notice that, from (9), β_{mv} is an increasing function thus confirming our tentative monotonicity conjecture. Our next result shows that low types submit lower bids under mean-variance preferences, while high types submit higher bids under mean-variance preferences than if they were to maximise expectations.

Proposition 1 (Single-crossing)**.** *Either $\beta_{mv}(\theta) \leq \beta_{rn}(\theta)$ for all $\theta \in [0, 1]$ or there exists a $\hat{\theta}$ in the support of F such that $\beta_{mv}(\theta) \leq \beta_{rn}(\theta)$ for $\theta \leq \hat{\theta}$, $\beta_{mv}(\hat{\theta}) = \beta_{rn}(\hat{\theta})$ and $\beta_{mv}(\theta) > \beta_{rn}(\theta)$ for $\theta > \hat{\theta}$.*

Proof. Proof of Proposition 1 Note that the symmetric equilibrium strategy can be written as

$$\beta_{\mathsf{mv}}(\theta) = \beta_{\mathsf{rn}} - \nu \int_0^{\theta} G(\vartheta) H(\vartheta) d\vartheta,$$

where $G(\vartheta) = \vartheta^2 (n-1)(F(\vartheta))^{n-2} f(\vartheta)$ and $H(\vartheta) = 1 - 2(F(\vartheta))^{n-1}$. F is a cumulative distribution function with density f, therefore $G(\vartheta) \geq 0$ for all $\vartheta \in [\underline{\theta}, \bar{\theta}]$. $H(\vartheta)$ is a continuous and decreasing function with $H(\underline{\theta}) = 1$ and $H(\bar{\theta}) = -1$. Hence, $\int_0^{\theta} G(\vartheta) H(\vartheta) d\vartheta > 0$ for sufficiently small $\theta > 0$ and if $\int_0^{\bar{\theta}} G(\vartheta) H(\vartheta) d\vartheta = 0$ for any $\hat{\theta} > 0$, then $\int_0^{\theta} G(\vartheta) H(\vartheta) d\vartheta < 0$ for all $\theta > \hat{\theta}$. ☐

This result is qualitatively in line with Propositions 1 and 2 in Fibich, Gavious, and Sela (2006). The intuition is that low-valuation bidders expect to lose in a symmetric equilibrium and therefore decrease their bids in order to keep their variance low. High-valuation bidders, by contrast, are likely to win and therefore increase their bids in line with variance compression. Proposition 1 says that there is only a single type of bidder endowed with mean-variance preferences who behaves in exactly the same way as her risk-neutral counter part.[10]

Corollary 1. *As the number of participating bidders n expands,*

1. *the convexity of the bidding function $\beta_{mv}(\theta)$ increases, i.e., low types decrease their bids and high types increase their bids relative to the case with a lower number of bidders;*

[10] This distortion of bids seems to correspond to experimental evidence. Both Barut, Kovenock, and Noussair (2002) and Noussair and Silver (2006) report bidding behaviour along these lines in all-pay auctions with private valuations. Moreover, gender differences in competitions à la Gneezy, Niederle, and Rustichini (2003) can be explained by our (a)symmetric risk-aversion result.

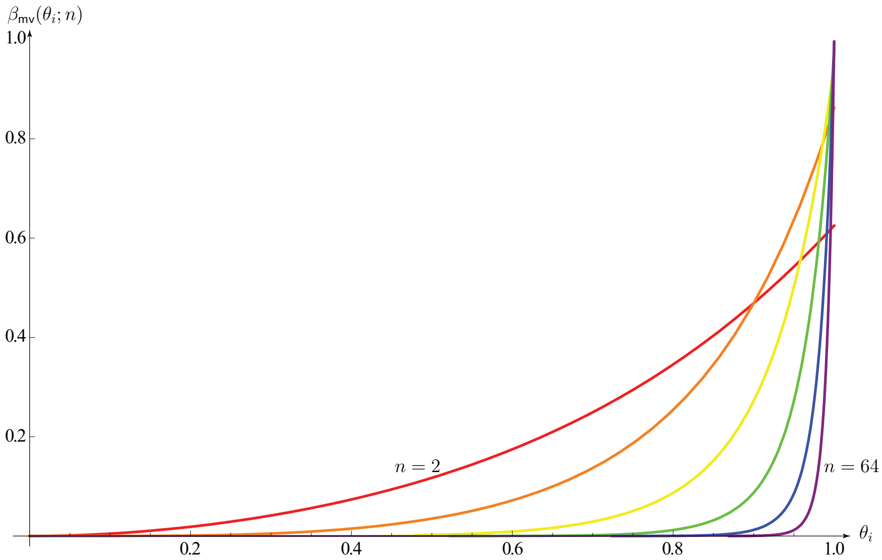


Figure 1: Equilibrium bidding functions for the all-pay auction under mean-variance preferences for uniformly distributed types, $\nu = 3/4$, and $n \in \{2, 4, 8, 16, 32, 64\}$ players, respectively (sorted in the colours of the rainbow from red to violet).

2. *the type $\hat{\theta}$ who issues the same bid under mean-variance and risk-neutral von Neumann-Morgenstern preferences shifts to the right.*

Proof. Proof of Corollary 1 Consider the derivative of (9) with respect to n

$$\theta f(\theta)F(\theta)^{n-3}\left[(1-\theta\nu)F(\theta)(1-\kappa)+2\theta\nu F(\theta)^{n}(1-2\kappa)\right] \tag{11}$$

where $\kappa = -(n-1)\log(F(\theta))$. Notice that $\log(F(\theta)) \leq 0$ and $\log(F(\theta))$ is strictly increasing in θ with $\log(F(\theta)) \to -\infty$ as θ approaches the lower bound of the support of its distribution and $\log(F(\theta)) \to 0$ as θ approaches the upper bound of the support of its distribution. Therefore, for sufficiently small θ, (11) becomes negative. Similarly, for θ sufficiently large, (11) is positive. □

The observation in the first part of corollary 1 qualitatively also holds with expected payoff maximizing players. The second part, however, shows that the effect is stronger when players are variance-averse.

3.1.1 Examples

The uniform distribution.

In the following, we exemplarily illustrate our findings for the case of n players when values are drawn from a uniform distribution over the interval $[0, 1]$. In this case, the expression for the objective of a bidder with mean-variance preferences simplifies to

$$u_i(b,\theta_i) = \theta_i(\beta^{-1}(b))^{n-1}(1-\nu\theta_i+\nu\theta_i(\beta^{-1}(b))^{n-1}) - b \tag{12}$$

which determines the symmetric equilibrium bidding functions as

$$b^* = \beta(\theta_i) = \frac{n-1}{n}\theta_i^n + \nu\left(\frac{n-1}{n}\theta_i^{2n} - \frac{n-1}{n+1}\theta_i^{n+1}\right) + C \tag{13}$$

for some constant C which is zero because a type-0 will not make a positive bid. Figure 2 compares this equilibrium bidding behaviour with that under standard risk-neutral von Neumann-Morgenstern preferences for two players. As seen before, the bidding behaviour of low-intermediate valuation types is more aggressive under expected payoff maximisation while high types submit higher bids under mean-variance preferences.

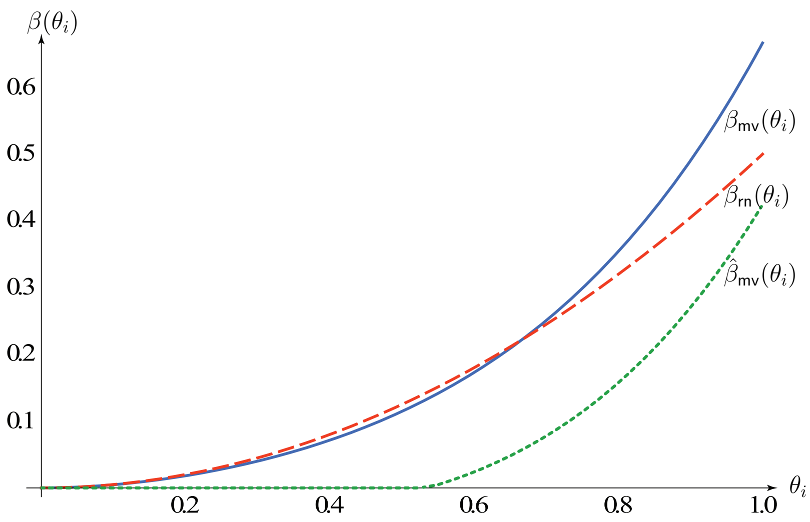


Figure 2: Equilibrium bidding functions for the all-pay auction under risk-neutrality (dashed, $\beta(\theta_i) = \theta_i^2/2$) and mean-variance preferences ($\nu = 1$, solid). The dotted bidding function results under mean-variance preferences if the prize itself is risky $\hat{\varepsilon} - \hat{\delta} = \frac{1}{4}$.

Other distributions.

Table 3.1.1 displays the symmetric, two-player equilibrium bidding functions for the most commonly used distribution functions when players have mean-variance preferences with $\nu = 1$ and values are i.i.d. according to the specified distribution.

Two-player equilibrium bidding functions with $\nu = 1$.

Distribution	$F(\theta)$	$f(\theta)$	$\beta(\theta)$
Uniform[0,1]	θ	1	$\frac{\theta_i^2}{2} - \frac{\theta_i^3}{3} + \frac{\theta_i^4}{2}$
Power[0,1]	θ^α	$\alpha\theta^{\alpha-1}$	$\frac{\alpha\theta_i^{1+\alpha}\left(2+\alpha-\theta_i-\alpha\theta_i+(2+\alpha)\theta_i^{1+\alpha}\right)}{(1+\alpha)(2+\alpha)}$
Beta(2,2)	$\frac{\int_0^\theta u(1-u)du}{\int_0^1 u(1-u)du}$	$\frac{\theta(1-\theta)}{\int_0^1 u(1-u)du}$	$2\theta_i^3 - 3\theta_i^4 + \frac{6\theta_i^5}{5} + 6\theta_i^6 - \frac{60\theta_i^7}{7} + 3\theta_i^8$
Quadratic-U	$4(\theta - {}^1\!/_2)^3 + 4({}^1\!/_2)^3$	$12(\theta - {}^1\!/_2)^3$	$\frac{3\theta_i^2}{2} - 5\theta_i^3 + \frac{21\theta_i^4}{2} - 24\theta_i^5 + 40\theta_i^6 - \frac{240\theta_i^7}{7} + 12\theta_i^8$

It is easy to find examples of distributions with unbounded support in which the equilibrium we derive above exists. For instance, the exponential distribution $F(\theta) = 1 - \exp(-\lambda\theta)$ gives rise to

$$\beta(\theta) = \frac{e^{-2\theta_i\lambda}\left(1 + e^{2\theta_i\lambda}(3 + 2\lambda) + 2\theta_i\lambda(1 + \theta_i\lambda) - 2e^{\theta_i\lambda}(2 + \lambda(1 + \theta_i(2 + \lambda + \theta_i\lambda)))\right)}{2\lambda^2}. \quad (14)$$

For general results over unbounded support, however, we would need to individually ensure that (9) is increasing and further restrict our sufficient condition in footnote 9. Since this leads to relatively inelegant analytic formulations we contend ourselves here with the compact support case.

3.2 Exogenous shocks

We want to motivate the analysis of exogenous noise with a separate stylised example. Consider an R&D company engaging in costly research outlays in order to obtain some (patentable) innovation first among a group of competitors. The endogenous variance is grounded, as before, in the uncertain spread between certain outlays and probabilistic winning. The exogenous component may be seen as market uncertainty in case of winning: the firm cannot usually be entirely certain about the market perception and success of its future product.

We now extend the basic model analysed in section 3 with exogenous noise. Consider expected revenue distributed $\mathbb{E}[\mathsf{R}] \sim W[\theta_i, \hat{\varepsilon}^2]$, where the distribution W is elliptical, i.e., completely determined by mean θ_i and variance $\hat{\varepsilon}^2 \in [0,1]$.[11] Similarly, we allow for the case that a player's valuation, if she does not win the object, is subject to another exogenous shock $\delta \sim L(0, \hat{\delta}^2)$. In the case of R&D competition, $\hat{\delta}^2$ reflects the uncertainty in the company's forecast of the residual demand after the innovation.

A competitor's objective therefore changes from (1) into

$$u_i(\mu(b,\theta_i), \sigma^2(b,\theta_i)) = \mu(b,\theta_i) - \nu\left(\sigma^2(b,\theta_i) + \hat{\varepsilon}^2 F(\beta^{-1}(b))^{n-1} + \hat{\delta}^2(1 - F(\beta^{-1}(b))^{n-1})\right) \quad (15)$$

for $\nu \in [0,1]$. Inserting back the expressions for the mean and variance (4), her objective is

$$\begin{aligned} u_i(b,\theta_i) = {} & (F(\beta^{-1}(b)))^{n-1}\left[\theta_i\left(1 - \nu\theta_i + \nu\theta_i(F(\beta^{-1}(b)))^{n-1}\right) - \nu\hat{\varepsilon}^2\right)\right] \\ & + (1 - (F(\beta^{-1}(b))))^{n-1}\hat{\delta}^2 - b. \end{aligned} \quad (16)$$

The first-order condition for maximisation of this expression with respect to b is

$$\begin{aligned} & \left[\theta_i\left(1 - \nu\theta_i + 2\nu\theta_i(F(\beta^{-1}(b)))^{n-1}\right) - \nu(\hat{\varepsilon}^2 - \hat{\delta}^2)\right] \times \\ & (n-1)(F(\beta^{-1}(b)))^{n-2} f(\beta^{-1}(b))\frac{\partial\beta^{-1}(b)}{\partial b} = 1. \end{aligned} \quad (17)$$

In the symmetric equilibrium $b = \beta(\theta_i)$ for all $i \in \mathcal{N}$, this yields the first-order differential equation

$$\begin{aligned} \beta'(\theta_i) = {} & \theta_i\left(1 - \nu\theta_i + 2\nu\theta_i(F(\theta_i))^{n-1}\right)(n-1)(F(\theta_i))^{n-2} f(\theta_i) \\ & - (n-1)(F(\theta_i))^{n-2} f(\theta_i)\nu(\hat{\varepsilon}^2 - \hat{\delta}^2). \end{aligned} \quad (18)$$

This differential equation is solved by the bidding function[12]

$$\hat{\beta}_{\mathsf{mv}}(\theta_i) = \begin{cases} 0 & \text{if } \theta_i \leq \theta_0 \\ \beta_{\mathsf{mv}}(\theta_i) - (F(\theta_i))^{n-1}\nu(\hat{\varepsilon}^2 - \hat{\delta}^2) & \text{if } \theta_i > \theta_0 \end{cases} \quad (19)$$

[11] Elliptical distributions are a generalisation of the normal family containing, among others, the uniform, Student-t, Logistic, Laplace, symmetric stable, and Normal distributions. A detailed presentation of these distributions is available in Fang, Kotz, and Ng (1987).

[12] Note that the bidding function (19) constitutes an equilibrium in both cases, whether a zero bid is interpreted as abstaining from the contest and hence results in a winning probability of zero, or the winner is determined by a tie-breaking rule in the event that all players bid zero, which happens with strictly positive probability if $\hat{\varepsilon}^2 - \hat{\delta}^2 > 0$ and F has full support $[0,1]$. Which of these cases is more appropriate depends on the exact environment to be modeled.

where $\beta_{\mathsf{mv}}(\theta_i)$ is defined in (10) and the 'cutoff type' θ_0 is implicitly defined as the solution to

$$\begin{aligned}&\theta_0(F(\theta_0))^{n-1} - \int_0^{\theta_0} (F(\vartheta))^{n-1} d\vartheta \\ &-\nu \int_0^{\theta_0} \vartheta^2 (n-1)(F(\vartheta))^{n-2} f(\vartheta) \left(1 - 2(F(\vartheta))^{n-1}\right) d\vartheta - (F(\theta_0))^{n-1}\nu(\hat{\varepsilon}^2 - \hat{\delta}^2) = 0\end{aligned} \tag{20}$$

for which a closed form solution is generally unavailable. As we restrict bids to be non-negative, the resulting bidding function is still invertible over the relevant region. Similarly to the common practice of normalising the valuation of the outside option to zero, (19) shows that there is a degree of freedom to normalise the variance of one of the two possible outcomes. In the remainder we therefore normalise $\hat{\delta}^2 \equiv 0$ for simplicity.

Corollary 2. *Introducing exogenous noise $\hat{\varepsilon}^2 - \hat{\delta}^2 > 0$ on the prize rotates the optimal bidding schedule down, causing low-type bidders to abstain from participating in the auction.*

Consequently, it lies in the interest of an effort maximizing contest designer to minimize $\hat{\varepsilon}^2 - \hat{\delta}^2$, i.e. reveal much information about the prize, while possibly keeping the loser's payoff uncertain.

3.2.1 Example

We round off this section with our usual uniform, two-bidders example. Consider the equilibrium bidding function

$$b^* = \hat{\beta}(\theta_i) = \frac{\theta_i^2}{2} - \nu\left(\frac{\theta_i^3}{3} - \frac{\theta_i^4}{2} + \theta_i\hat{\varepsilon}^2\right). \tag{21}$$

The bidding behaviour this suggests for the case of a stochastic prize parameterised by $\hat{\varepsilon}^2 = {}^1\!/_4$, is shown as dotted line in figure 2. Consider now a case in which we auction two objects valued $\theta_1 > \theta_2$ with exogenous prize variance $\hat{\varepsilon}^1 > \hat{\varepsilon}^2$. If (full demand) bidders submit separate bidding functions for each object, then we can get cases where the bid for the high-value/high-risk object is below that of the low-value/low-risk object. An example under uniform valuations and $\nu = 1$ is $\beta(\theta_1 = 3/4|\hat{\varepsilon}_1^2 = 1/4) = 0.111 < 0.139 = \beta(\theta_2 = 2/3|\hat{\varepsilon}_2^2 = 1/8)$.

3.3 Two asymmetric bidders

This section presents our results on all-pay auctions between bidders who are not identical in terms of their risk preferences. Consider the following uniform, two-players setup featuring asymmetric degrees of variance aversion ν_i where player $i \in \{1, 2\}$ maximises

$$u_i(\mu(b,\theta_i), \sigma^2(b,\theta_i)) = \mu(b,\theta_i) - \nu_i\sigma^2(b,\theta_i), \ \ \nu_i \in \mathbb{R}_+. \tag{22}$$

We consider the particular case of $\nu_1 = 0$ and $\nu_2 = \nu$, i.e., bidder 1 is risk-neutral while bidder 2 is variance-averse. Therefore, we get

$$\begin{aligned} u_1(\theta_1, b_1) &= \beta_2^{-1}(b_1)\theta_1 - b_1, \\ u_2(\theta_2, b_2) &= \beta_1^{-1}(b_2)\theta_2 - \nu\left(\theta_2^2\beta_1^{-1}(b_2)(1-\beta_1^{-1}(b_1))\right) - b_2 \end{aligned} \tag{23}$$

with the pair of first-order conditions

$$\begin{aligned}\frac{\partial u_1(\theta_1,b_1)}{\partial b_1} &= \frac{1}{\beta_2'(\beta_2^{-1}(b_1))}\theta_1 - 1 = 0\\ &\Leftrightarrow \beta_2'(\beta_2^{-1}(b_1)) = \theta_1, \qquad (24)\\ \frac{\partial u_2(\theta_2,b_2)}{\partial b_2} &= \frac{1}{\beta_1'(\beta_1^{-1}(b_2))}\left(\theta_2 - \nu\theta_2^2(1-2\beta_1^{-1}(b_2))\right) - 1 = 0\\ &\Leftrightarrow \beta_1'(\beta_1^{-1}(b_2)) - 2\nu\theta_2^2\beta_1^{-1}(b_2) = \theta_2 - \nu\theta_2^2. \qquad (25)\end{aligned}$$

In equilibrium, $b_1 = \beta_1(\theta_1)$ and $b_2 = \beta_2(\theta_2)$. Thus, we substitute $\beta_1^{-1}(b_1) = \theta_1$ into (24) to obtain

$$\beta_2'(\beta_2^{-1}(b)) = \beta_1^{-1}(b). \qquad (26)$$

Taking the derivative of $\beta_1^{-1}(b)$ and applying (24) gives

$$\beta_1'(\beta_1^{-1}(b)) = \frac{\beta_2'(\beta_2^{-1}(b))}{\beta_2''(\beta_2^{-1}(b))} \qquad (27)$$

where we use b as variable from the joint support of $\beta_1(\cdot)$ and $\beta_2(\cdot)$.[13] Substituting (27) and (26) into (25) yields the following second-order differential equation in β_2

$$\frac{\beta_2'(\beta_2^{-1}(b))}{\beta''_2(\beta_2^{-1}(b))} - 2\nu\theta^2\beta_2'(\beta_2^{-1}(b)) = \theta - \nu\theta^2. \qquad (28)$$

This differential equation can be solved using the boundary condition $\beta_2(0) = 0$ to obtain

$$\begin{aligned}\beta_2(\theta_2) &= \frac{1}{2\sqrt{1+4c}\nu}\Big[\sqrt{1+4c}\left(-1+\theta\nu+\sqrt{1+\theta\nu(-2+\theta\nu+4c\theta\nu)}+\log(2)\right)\\ &+ \log\left(1-\sqrt{1+4c}\right) - \sqrt{1+4c}\log\left(1-\theta\nu+\sqrt{1+\theta\nu(-2+\theta\nu+4c\theta\nu)}\right) \qquad (29)\\ &- \log\left(1-\nu\left(\theta+4c\theta+\sqrt{1+4c}\sqrt{\tfrac{1}{\nu^2}+\tfrac{\theta}{\nu}(-2+\theta\nu+4c\theta\nu)}\right)\right)\Big]\end{aligned}$$

for yet undetermined constant of integration c. In order to solve for the first player's bidding function, we solve (26) for

$$\beta_1(\theta) = \beta_2\left((\beta_2')^{-1}(\theta)\right) \qquad (30)$$

where

$$(\beta_2')^{-1}(\theta) = \frac{\theta}{\nu(\theta-\theta^2+c)}. \qquad (31)$$

From an argument similar to the one used in a standard (risk-neutral) all-pay auction follows that the two bidding functions $\beta_1(\cdot)$ and $\beta_2(\cdot)$ must share the same support. Intuitively, in equilibrium, no player's type can submit a strictly higher bid than the other player's highest type. Setting $\beta_1(1) = \beta_2(1)$ implies that the only possible value for the constant of integration is

$$c = \frac{1}{\nu}. \qquad (32)$$

[13] The standard argument applies that in the two-player, all-pay auction the supports of both players' bidding functions coincide.

Substituting this constant into (29), we obtain the following pair of bidding functions

$$
\begin{aligned}
\beta_1(\theta) &= \frac{\log(1-\theta^2\nu+\theta\nu)}{2\nu} - \frac{\theta^2}{\theta\nu(\theta-1)-1} \\
&+ \frac{1}{2\sqrt{\nu(4+\nu)}}\left(\log\left(1-\sqrt{\tfrac{4+\nu}{\nu}}\right) - \log\left(\frac{\sqrt{\frac{4+\nu}{\nu}}+\theta\left(4+\theta\left(\nu+\sqrt{\nu(4+\nu)}\right)\right)-1}{\theta\nu(\theta-1)-1}\right)\right), \\
\beta_2(\theta) &= \frac{1}{2\nu}\left[\theta\nu-1+\sqrt{1-2\theta\nu+\theta^2\nu(4+\nu)}+\sqrt{\tfrac{\nu}{4+\nu}}\log\left(\sqrt{\tfrac{4+\nu}{\nu}}-1\right)\right. \\
&- \log\left(\tfrac{1}{2}\left(1-\theta\nu+\sqrt{1-2\theta\nu+\theta^2\nu(4+\nu)}\right)\right) \\
&- \left.\sqrt{\tfrac{\nu}{4+\nu}}\log\left(\theta(4+\nu)-1+\sqrt{\tfrac{(4+\nu)(1-2\theta\nu+\theta^2\nu(4+\nu))}{\nu}}\right)\right],
\end{aligned} \tag{33}
$$

which are illustrated for the case of $\nu = 1$ in figure 3.

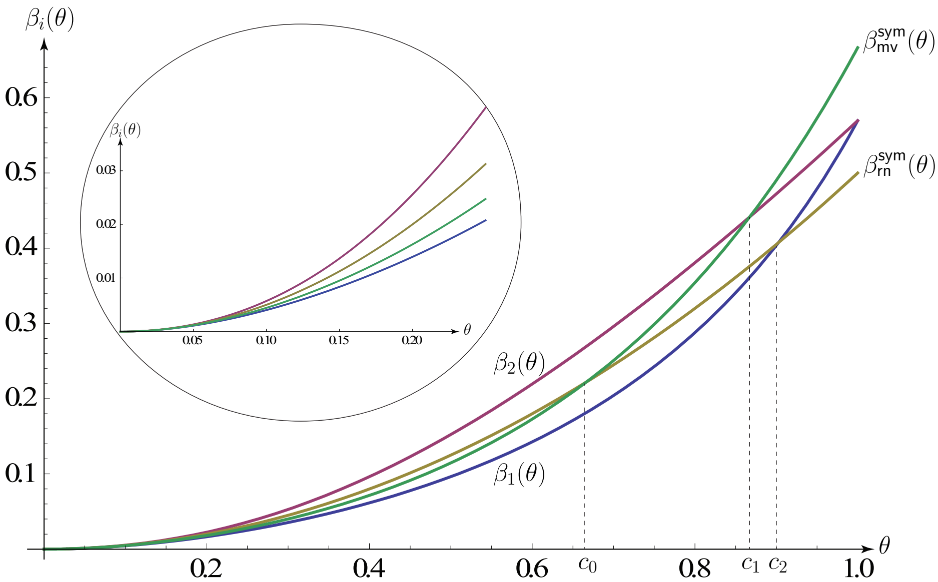


Figure 3: Comparison of asymmetric and symmetric bidding under mean-variance preferences for $\nu = 1$.

As the figure shows, each positive risk-neutral player type bids less than the corresponding type of her variance-averse opponent. While the ν-variance-averse, asymmetric bidder with bidding function $\beta_2(\cdot)$ always bids more than symmetric risk-neutral bidders $\beta_{\text{rn}}^{\text{sym}}$, the asymmetric risk-neutral bidder with bidding function $\beta_1(\cdot)$ (competing with a variance-averse player) bids up to a cutoff-type c_2 below the symmetric risk-neutral bidders and, for types higher than c_2, she bids above. Similarly, the asymmetric variance-averse bidder (competing with a risk-neutral bidder) bids up to a cutoff-type c_1 above the symmetric ν-variance-averse bidders ($\beta_{\text{mv}}^{\text{sym}}$) and bids below for types higher than c_1. Both properties are qualitatively similar to the single-crossing property with cutoff $\hat{\theta} = c_0$ from proposition 1. The generally high bids of the variance-averse bidder cause low types of the risk-neutral bidder to bid less in comparison to their strategy when faced with risk-neutral opponents. High types of the risk-neutral bidder, on the other hand, increase their bid in reaction to their variance-averse opponent's strategy.

4 Revenue valuation

The classical reference for revenue valuation in winner-pay auctions under risk aversion is Holt (1980) who discusses a procurement setup. Revenue equivalence between the standard auction formats breaks down with risk averse bidders. While second-price bidders maintain their dominant strategies of bidding their values, first-price competitors increase their bids with respect to the standard, risk-neutral case. This is due to the fact that raising one's bid in a first-price auction can be seen as partial insurance against loosing. From a risk averse seller's point of view, the first-price auction is preferable to a second-price format because it exposes the seller to less revenue risk.[14]

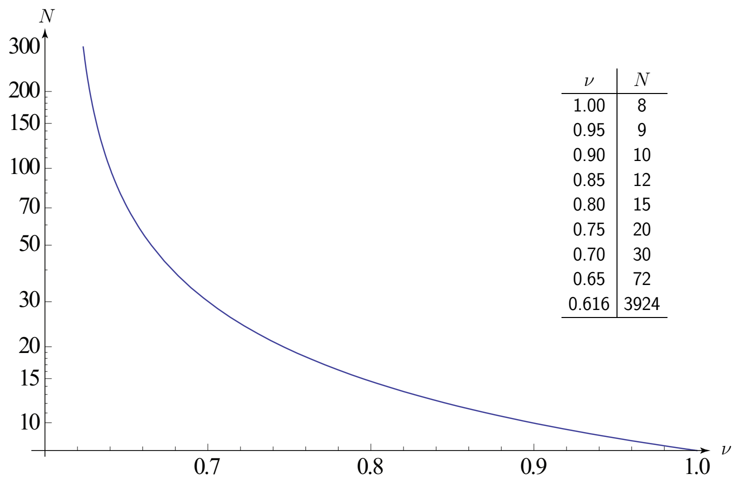
In this section we limit attention to uniformly distributed bidder valuations because our objective lies in the derivation of a series of concrete revenue ranking results. The results, however, are qualitatively similar for the other distributions listed in the table of section 3.1.1. The seller's expected revenue R depends on the bidder's preferences. In the case of risk-neutral bidders with von Neumann-Morgenstern preferences, the seller expects to earn

$$\mathbb{E}[\mathsf{R}_{\mathsf{rn}}] = n\int_0^1 \frac{n-1}{n}\theta^n \, d\theta = \frac{n-1}{n+1}. \tag{34}$$

If the bidders exhibit mean-variance preferences, then the seller can expect

$$\begin{aligned} \mathbb{E}[\mathsf{R}_{\mathsf{mv}}] &= n\int_0^1 \frac{n-1}{n}\theta^n + \nu\left(\frac{n-1}{n}\theta^{2n} - \frac{n-1}{n+1}\theta^{n+1}\right) d\theta \\ &= \frac{n-1}{n+1} + \nu\left(\frac{n-1}{2n+1} - \frac{n(n-1)}{(n+1)(n+2)}\right). \end{aligned} \tag{35}$$

The revenue limit for $n \to \infty$ is $1 - \nu/2$. This limit, however, is only approached from below for low values of ν. As figure 5 illustrates graphically, for high variance weights ν, there exists a revenue-maximal finite number of bidders N. The following corollary states this revenue-optimal exclusion result and figure 4 gives a graphical illustration.



ν	N
1.00	8
0.95	9
0.90	10
0.85	12
0.80	15
0.75	20
0.70	30
0.65	72
0.616	3924

Figure 4: Expected revenue maximising numbers of participants n (on a logarithmic scale) as a function of the variance aversion parameter ν.

[14] For references, see Milgrom (2004, p123).

Corollary 3. *An expected revenue $\mathbb{E}[R_{mv}]$ maximising seller finds it optimal to limit the number of participants in an all-pay auction if bidders have a sufficiently high variance aversion parameter ν. This optimal number of participants N is decreasing in ν.*

Holding the number of players fixed, expected revenue is strictly increasing in ν for the two-players case and strictly decreasing in ν for all $n > 2$. This is illustrated in figure 5 and summarised in corollary 4.

Corollary 4. *For $n = 2$, the expected revenue is strictly greater if bidders exhibit mean-variance preferences than if they are expected payoff maximisers. For all other n, this relationship is reversed.*

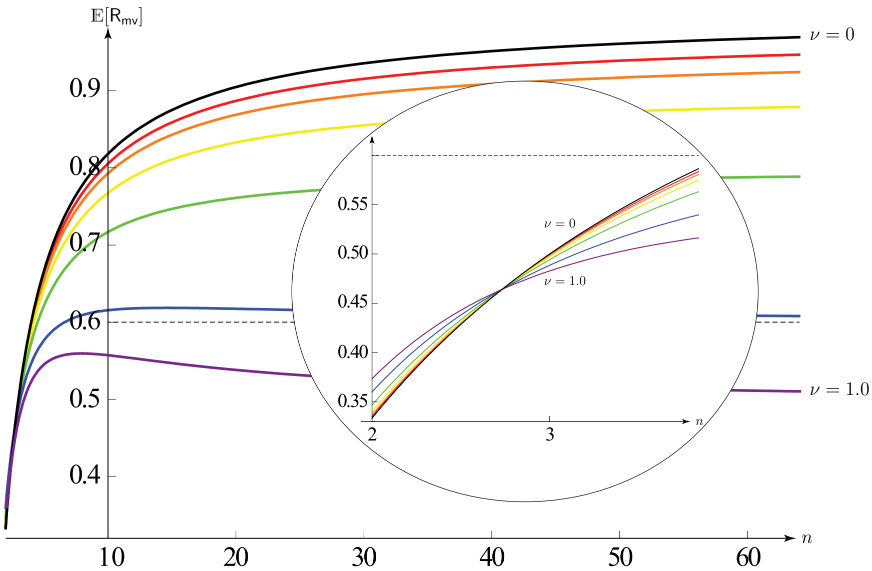


Figure 5: Revenue for $\nu \in \{.05, .1, .2, .4, .8, 1.0\}$ players, respectively (sorted in the colours of the rainbow from red to violet) and $\nu = 0$ in black for $n \in [2, 64]$.

If the seller himself also considers the revenue variance in addition to the revenue's mean, then his preference may be reversed. In the case of risk-neutral bidding, the seller's revenue variance is

$$\begin{aligned} \mathbb{V}[\mathsf{R}_{\mathsf{rn}}] &= n \int_0^1 \left(\beta_{\mathsf{rn}}(\theta) - \frac{\mathbb{E}[\mathsf{R}_{\mathsf{rn}}]}{n} \right)^2 d\theta \\ &= \frac{n(n-1)^2}{(2n+1)(n+1)^2}. \end{aligned} \tag{36}$$

The bidders behaving according to mean-variance preferences cause a revenue variance of

$$\begin{aligned} \mathbb{V}[\mathsf{R}_{\mathsf{mv}}] &= n \int_0^1 \left(\beta_{\mathsf{mv}}(\theta) - \frac{\mathbb{E}[\mathsf{R}_{\mathsf{mv}}]}{n} \right)^2 d\theta \\ &= \frac{n(n-1)^2}{(1+2n)^2} \left(\frac{1+2n}{(1+n)^2} + \nu \frac{7+n(21-4n(n-3))}{(1+n)^2(2+n)(1+3n)} \right. \\ &\qquad \left. + \nu^2 \frac{74+n(151+8n(n-3)(n-1))}{(2+n)^2(3+2n)(2+3n)(1+4n)} \right). \end{aligned} \tag{37}$$

The rate of $\mathbb{V}[\mathsf{R}_{\mathsf{mv}}]/\mathbb{V}[\mathsf{R}_{\mathsf{rn}}]$ is given by

$$1 + \frac{(7+(7-2n)n)\nu}{(2+n)(1+3n)} + \frac{(1+n)^2(74+n(151+8(n-3)(n-1)n))\nu^2}{(2+n)^2(1+2n)(3+2n)(2+3n)(1+4n)}. \tag{38}$$

For $n = 2, 3, 4$, this ratio is greater than one and increasing in ν. For $n \geq 5$ the variance ratio is decreasing in ν and below one. This implies the following corollary.

Corollary 5.

	$n = 2$	$n = 3, 4$	$n \geq 5$
$\mathbb{E}[R]$	$\mathbb{E}[R_{mv}] > \mathbb{E}[R_{rn}]$	$\mathbb{E}[R_{mv}] < \mathbb{E}[R_{rn}]$	$\mathbb{E}[R_{mv}] < \mathbb{E}[R_{rn}]$
$\mathbb{V}[R]$	$\mathbb{V}[R_{mv}] > \mathbb{V}[R_{rn}]$	$\mathbb{V}[R_{mv}] > \mathbb{V}[R_{rn}]$	$\mathbb{V}[R_{mv}] < \mathbb{V}[R_{rn}]$

Therefore, for $n = 3, 4$, a seller with both types of preferences will prefer bidders maximising expected payoff. In all other cases, a variance-averse seller may prefer bidders with mean-variance preferences, where the exact ranking depends on the degree of the seller's variance aversion.

4.1 Optimal reserve prices

Corollary 4 shows that with only two bidders the expected revenue of an all-pay auction is strictly increasing in the degree of variance aversion, although the opposite relationship is true for any other number of players. We now introduce an exogenous reserve price $p_r > 0$ into our revenue analysis to show that this ordering can be reversed by choosing an optimal reserve price.[15]

In the symmetric equilibrium, either one of two symmetric, variance-averse players will participate in the auction if their utility at bidding p_r equals

$$u_i(\mu(b = p_r, \theta_i), \sigma^2(b = p_r, \theta_i)) = \mu(p_r, \theta_i) - \nu\sigma^2(p_r, \theta_i) = 0, \tag{39}$$

where

$$\mu(p_r, \theta_i) = \theta_i\theta_i - p_r, \text{ and } \sigma^2(p_r, \theta_i) = \theta_i(\theta_i(1 - \theta_i))^2 + (1 - \theta_i)(\theta_i\theta_i)^2. \tag{40}$$

The first participating type in the contest with reserve price p_r, θ_r who solves (39) is implicitly defined by

$$p_r = \theta_r^2 + (\theta_r - 1)\theta_r^3\nu. \tag{41}$$

The solution to this equation, $\theta_r^{-1}(p_r)$, gives this type as a function of the reserve price.[16] Only if a player's type θ_i is at least as high as $\theta_r^{-1}(p_r)$, she will participate in the auction. Her maximisation problem (16) gives the bidding function $\beta_r(\theta)$ as equivalent of (19) as solution to

$$\beta_r(\theta) = \int_{\theta_r^{-1}(p_r)}^{\theta} \vartheta(1 - \nu\vartheta + 2\nu\vartheta^2)\, d\vartheta + p_r. \tag{42}$$

The seller's expected revenue when setting reserve price p_r is now

$$\mathbb{E}[\mathsf{R}_{\mathsf{mv}}(p_r)] = 2\int_{\theta_r^{-1}(p_r)}^{1} \beta_r(\vartheta)\, d\vartheta; \tag{43}$$

it is shown for various ν in figure 6. As evident from the figure, the revenue-maximising reserve price p_r^* is lower when players are more variance-averse.

[15] Whether sellers actually set reserve prices optimally is debatable, Davis, Katok, and Kwasnica (2011) investigate this question in the context of winner-pay auctions in a laboratory experiment and find amongst other possible explanations that risk-aversion can explain parts of the observed data.

[16] As the explicit form of $\theta_r^{-1}(p_r)$, (49), is rather unappealing it is relegated to the appendix (as are all following expressions which are based on it).

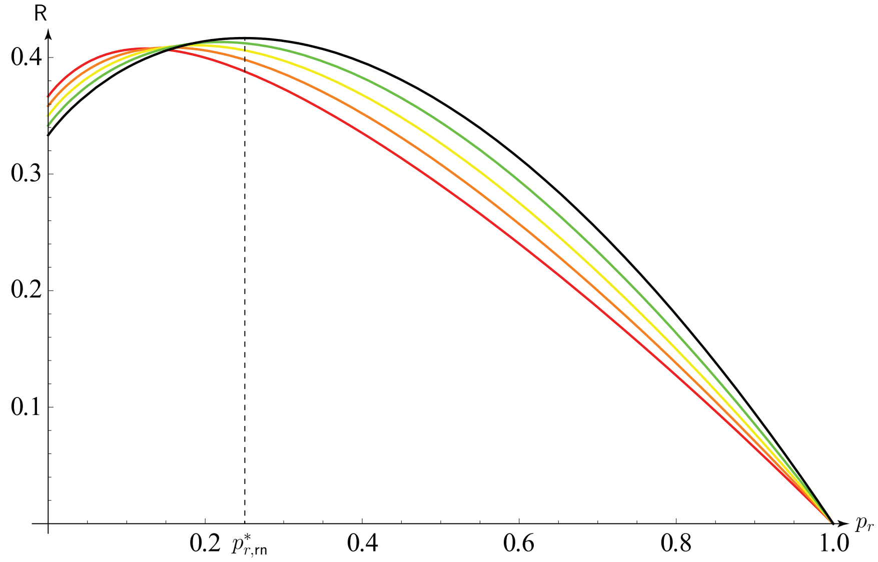


Figure 6: Seller's expected revenue (in the colours of the rainbow from red $\nu = 1.0$ in steps of $.25$ to green $\nu = 0.25$) as a function of the reserve price p_r under variance-averse bidding compared to the case of risk-neutral bidding $\nu = 0$ (black).

Corollary 6. *The highest expected revenue achievable by optimally setting a reserve price in a symmetric two-player all-pay auction is decreasing in the degree of variance aversion. The revenue maximizing reserve prize, p_r^*, is decreasing in the degree of bidders' variance aversion, ν.*

5 Concluding remarks

We present first results for the study of all-pay auctions if buyers or sellers are endowed with mean-variance preferences. We fully characterise the symmetric equilibrium bidding functions of the all-pay auction with n identical bidders when bidders maximise an additively separable function of their expected payoff and payoff variance. Our first proposition shows that consideration of mean-variance preferences suffices to derive the qualitative properties of the bidding function which Fibich, Gavious, and Sela (2006) obtain in their analysis of a similar environment but considering any von Neumann-Morgenstern utility function which entails risk aversion. These qualitative properties seem to correspond well to experimental data. As such, all our results based on this bidding function appear relevant even when payoff distributions are not fully characterized by their first two moments.

In our model of mean-variance preferences, players choose a strategy that maximises the difference between their expected payoff and the payoff variance, which is weighted by a parameter, ν, representing the players' degree of variance aversion. One major advantage of this approach is that we obtain closed form solutions for the bidding functions with just a single parameter representing risk aversion. Thus, we can perform comparative statics. Furthermore, this functional form allows us to relax the standard assumption of identical preferences. We exemplary solve for the bidding functions in an all-pay auction with one expected payoff maximiser and one bidder with mean-variance preferences. In contrast to the symmetric equilibrium, we find that the mean-variance bidder of a given type always bids more than her risk neutral opponent of the same type. Although the analysis is only provided for the case of two bidders, the result would look similar if more general sets of n_1 risk neutral and n_2 mean-variance bidders were competing. Similarly, we conjecture that the qualitative findings from our benchmark case would carry over if the first bidder type was not risk neutral, but just less variance averse than her opponent.

Having obtained the (symmetric) equilibrium bidding function we then turn to the seller's perspective and consider effects of the number of bidders, their degree of variance aversion, and an optimally set reserve price. Corollary 4 shows that the influence of variance aversion on expected revenue depends on the number of players. In particular, we find that considering $n \geq 3$ reverses the ranking found for the two-player case. This finding suggests that under risk-aversion the generalisation from the two-player case to the general case may not always be as intuitive as it is often the case under risk neutrality. Furthermore, we find that the expected revenue is only increasing in the number of players as long as players are not too variance averse. If players exhibit a sufficiently high degree of variance aversion, then a seller would optimally want to limit the number of participants in the contest. One way of doing so could possibly be a multi stage sequential-elimination contest à la Fu and Lu (2012)—a mean-variance analysis of which is left for future research.

With the exception of the analysis of bidding behaviour of n ex-ante identical players, much of our analysis focuses on the case of valuations that are i.i.d. draws from the uniform distribution over $[0, 1]$. The resulting simplification of otherwise lengthy expressions and the possibility to analytically obtain solutions has caused us to make this assumption. However, qualitatively similar results can be obtained for other standard distributions.

Appendix

6 Second-order condition

The second-order condition is obtained by twice differentiating the objective (2) and supplying (9) for $\frac{\partial \beta^{-1}(b)}{\partial b}$ and $\frac{\partial^2 \beta^{-1}(b)}{\partial b^2} = -\frac{\beta''(\theta)}{\beta'(\theta)^3}$. The resulting expression simplifies to

$$F(\theta_i)^{5-2n} \frac{F(\theta_i)^2(\theta_i\nu - 2\theta_i\nu F(\theta_i) - 1)\left(1 - 2\theta_i\nu + 4\theta_i\nu F(\theta_i)^{n-1}\right) + \theta_i f(\theta_i) D}{(n-1)^2\theta_i^2 f(\theta_i)\left(F(\theta_i) - \theta_i\nu F(\theta_i) + 2\theta_i\nu F(\theta_i)^n\right)^3} \tag{44}$$

where

$$\begin{aligned} D = &(\theta_i\nu - 1)F(\theta_i)((n-2)(1-\theta_i\nu) + 2(n-3)\theta_i\nu F(\theta_i)) \\ &- 2\theta_i\nu F(\theta_i)^n((2n-3)(1-\theta_i\nu) + 4(n-2)\theta_i\nu F(\theta_i)), \end{aligned} \tag{45}$$

which, under the sufficient condition (7), is negative.

7 Explicit forms used in revenue derivation

This appendix shows the explicit form of some of the equations kept for presentation reasons from the main text. Define

$$A = \sqrt[3]{27p_r\nu^2 - 2 - 72p_r\nu + \sqrt{4(12p_r\nu - 1)^3 + (2 + 9p_r(8 - 3\nu)\nu)^2}}, \tag{46}$$

$$B = \sqrt{3 - \frac{8}{\nu} + \frac{4\ 2^{1/3}(12p_r\nu - 1)}{\nu A} - \frac{2\ 2^{2/3}A}{\nu}}, \tag{47}$$

and

$$C = \sqrt{3}B - 3 - \sqrt{6}\sqrt{3 - \frac{8}{\nu} + \frac{2\ 2^{1/3}(1 - 12p_r\nu)}{\nu A} + \frac{2^{2/3}A}{\nu} - \frac{3\sqrt{3}(\nu - 4)}{\nu B}}. \tag{48}$$

Then the inverse of (41) is given explicitly as

$$\theta_r^{-1}(p_r) = \frac{3 - \sqrt{3}B + \sqrt{6}\sqrt{3 - \frac{8}{\nu} + \frac{2\ 2^{1/3}(1-12p_r\nu)}{\nu A} + \frac{2^{2/3}A}{\nu} - \frac{3\sqrt{3}(\nu-4)}{\nu B}}}{12}. \tag{49}$$

The explicit version of (42) is

$$\beta_r(\theta_i) = \frac{3p_r + \theta_i^2\left(3 - 2\nu\theta_i + 3\nu\theta_i^2\right) + \nu\left(\frac{C}{12}\right)^3}{6}. \tag{50}$$

Finally, the explicit version of the revenue of the all-pay auction with two symmetrically variance-averse players with parameter ν and reserve price p_r (43) is

$$\mathsf{Rev}_{\mathsf{mv}}(p_r) = \frac{10 + 39p_r + \nu + 3p_rC + (4+\nu)\left(\frac{C}{12}\right)^3 - \left(\frac{C}{4}\right)^2}{30}. \tag{51}$$

In principle, the derivative of the last expression with respect to p_r gives an explicit version of the revenue-optimal reserve price p_r^*. This derivation is not shown here for reasons of economy of space.

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