Fairness in Auctions and Task Assignments

Duygu Yengin
Fairness in Auctions and Task Assignments*

Duygu Yengin†

September 8, 2009

Abstract

We study the problem of allocating objects when monetary transfers are possible. We are interested in mechanisms that allocate the objects in an efficient way and induce the agents to report their true preferences. Within the class of such mechanisms, first we characterize egalitarian-equivalent mechanisms. Then, we add a bounded-deficit condition and characterize the corresponding class. Finally, we investigate the relations between egalitarian-equivalence and other fairness notions such as no-envy.

JEL Classifications: C79, D61, D63.

Key words: allocation of indivisible goods and money, fair auctions, task assignments, strategy-proofness, the Vickrey-Clarke-Groves mechanisms, order preservation, egalitarian-equivalence, no-envy, egalitarianism.

1 INTRODUCTION

We study problems where a principal, which we call the “center”, has to allocate tasks among agents based on the costs the agents incur for performing the tasks. The center may be a social planner (government) which pursues goals like efficiency and fairness. In order to induce the agents to report their costs truthfully, the center must offer them incentives: money transfers are made between the center and the agents. Agents have preferences over the sets of tasks and transfers. We assume that preferences are represented by quasi-linear utility functions, all tasks must be allocated, each task is assigned to only one agent; and there is no restriction on the number of tasks or the size of the transfer an agent can be assigned.

Examples to this task assignment problem include job and wage assignments and imposition of tasks on agents. Specific cases of the last example are government requisitions and eminent domain proceedings (see Yengin, 2008a). However, our results can be easily extended to a more general setting where heterogenous desirable objects are allocated among a finite set of agents whose valuations for the objects are their private information and monetary transfers are allowed. Among the many real life examples are auctions, the allocation of donated goods and money among the needy, the allocation of inheritance among heirs, the allocation of landing rights to airlines, and the allocation of water entitlements, fishing and pollution permits.

A mechanism determines who is assigned which tasks and what the transfers are. Our aim is to design mechanisms that attain three essential goals: efficiency, immunity to manipulations, and equity.

*The first draft of this paper was written while I was a Ph.D. student at the University of Rochester. I am grateful to William Thomson for his guidance and advice. I also thank Paulo Barelli and Gábor Virág for their helpful comments and advice.

†School of Economics, The University of Adelaide, Napier Building, Room G 34, SA 5005, Australia; e-mail: duygu.yengin@adelaide.edu.au.
In terms of efficiency, we are interested in mechanisms that assign the tasks such that the total cost incurred by the agents is minimal\(^1\) (assignment-efficiency).

Agents may manipulate the allocation in their favor by misrepresenting their costs. Hence, an assignment-efficient mechanism can only minimize the actual total cost if the mechanism is immune to such manipulations. Strategy-proofness requires that truthful revelation of costs be a dominant strategy for all agents.

The equity concept we consider here, egalitarian-equivalence (Pazner and Schmeidler; 1978), is arguably, one of the main fairness notions that have been extensively studied in the fair allocation literature. Fairness is especially important when tasks are imposed on agents (against their will) as in eminent domain proceedings. Also, when agents have equal rights over the allocated objects, government is concerned about the fairness of the allocation. Examples to this case include allocation of donated goods among the needy, auctions held to allocate water licences among the farmers, allocation of pollution permits to factories and so on.

We have in mind situations where agents have equal rights over the resources or equal responsibilities over the completion of tasks. In cases like these, assigning each agent the same bundle might be seen as fair. However, in general, tasks are not identical and such an allocation composed of identical bundles may not be feasible. Then, an alternative way to ensure fairness is to choose a feasible allocation that is Pareto-indifferent to an identical-bundle allocation. A mechanism is egalitarian-equivalent if for each economy, it chooses a feasible allocation that leaves each agent indifferent between her assigned bundle and a common reference bundle composed of a reference set of tasks and a reference transfer.

Another perspective that would lead to egalitarian-equivalence as an equity concept is the liberal-egalitarian theory of justice:

Suppose every agent were assigned the same bundle. Then, the utility differences of agents would be solely due to the differences in their cost functions (i.e. preferences). If agents are held responsible for their costs (preferences), but not for the heterogeneity in the resources, then this allocation would be fair. Since an allocation composed of identical bundles may not exist, an alternative to support this liberal-egalitarian notion is to use an egalitarian-equivalent mechanism.\(^2\)

It is well known that in the class of problems we study, the so called Vickrey-Clarke-Groves mechanisms (simply referred to as the Groves mechanisms) are the only mechanisms satisfying the first two goals we want to achieve: assignment-efficiency and strategy-proofness. Adding equity as an additional requirement leads to our main result: characterization of the class of egalitarian-equivalent Groves mechanisms.

By Green and Laffont (1977), no Groves mechanism balances the budget. Still, it is possible to design egalitarian-equivalent Groves mechanisms that generate bounded budget deficits. Our second result characterizes the class of such mechanisms. Among the egalitarian-equivalent Groves mechanisms that respect the same upper bound on deficit, we specify the Pareto-dominant ones.

Finally, we analyze the relations between the egalitarian-equivalent Groves mechanisms and Groves mechanisms satisfying other fairness axioms, such as no-envy or order-preservation. We find out that under assignment-efficiency and strategy-proofness, interesting logical relations hold between several fairness axioms that do not exist otherwise.

The analysis of the Groves mechanisms from the fairness perspective has been the object of only few recent papers. In Yengin (2008a), we introduce the class of Super-Fair Groves

\(^1\)In case of desirable objects, the total value experienced by all agents should be maximized.

\(^2\)Egalitarian-equivalence can be related to the idea of “equality of resources” (Dworkin, 2000). For a more detailed philosophical motivation for egalitarian-equivalence based on the liberal-egalitarian distributive theory of justice, see Yengin (2008a).
mechanisms and characterize this class with several sets of fairness axioms (one of which is egalitarian-equivalence and no-envy). In the same model as ours, Pápai (2003) characterizes envy-free (no agent prefers another agent’s bundle to her own) Groves mechanisms. Porter, Shoham, and Tennenholtz (2004) introduce a class of Groves mechanisms that respect a welfare lower bound based on Rawl’s maximin equity criterion (k–fairness) and Atlamaz and Yengin (2008) characterize this class. Yengin (2008b) considers several welfare bounds (including the identical-preferences lower-bound) and characterizes Groves mechanisms that respect them.

In Section 2, we present the model and define the Groves mechanisms. In Section 3, we characterize egalitarian-equivalent Groves mechanisms. In Section 4, we investigate the implications of imposing an upper-bound on the deficit. Section 5 analyses the relations between different classes of Groves mechanisms. All proofs are in the Appendix.

2 Model

A finite set of indivisible tasks is to be allocated among a finite set of agents. All tasks must be allocated. An agent can be assigned either no task, a single task, or more than one task. Each task is assigned to only one agent. Let \( \mathbb{A} \) be the finite set of tasks, with \( |\mathbb{A}| \geq 1 \), and \( \alpha, \beta \) be typical elements of \( \mathbb{A} \).

There is an infinite set of “potential” agents indexed by the positive natural numbers \( \mathbb{N} = \{1, 2, \ldots\} \). In any given problem, only a finite number of them are present. Let \( \mathcal{N} \) be the set of subsets of potential agents with at least two agents. Let \( n \geq 2 \) and \( N \) with \( |N| = n \) be a typical element of \( \mathcal{N} \). The number of agents may be smaller than, equal to, or greater than the number of tasks.

Let \( 2^\mathbb{A} \) be the set of subsets of \( \mathbb{A} \). Each agent \( i \) has a cost function \( c_i : 2^\mathbb{A} \to \mathbb{R}_+ \) with \( c_i(\emptyset) = 0 \). \(^3\) Let \( \mathcal{C} \) be the set of such cost functions and \( \mathcal{C} = \bigcup_{N \in \mathcal{N}} \mathcal{C}^N \) where \( \mathcal{C}^N \) is the \( n \)-fold Cartesian product of \( \mathcal{C} \).

For each \( N \in \mathcal{N} \), a cost profile for \( N \) on the domain \( \mathcal{C} \) is a list \( c \equiv (c_1, \ldots, c_n) \) where for each \( i \in N \), \( c_i \in \mathcal{C} \). A cost profile defines an economy. Let \( c, c', \bar{c} \) be typical economies with associated agent sets \( N, N', \bar{N} \). For each \( N \in \mathcal{N} \) and each \( i \in N \), let \( c_{-i} \) be the cost profile of the agents in \( N \setminus \{i\} \).

There is a perfectly divisible good we call “money”. Let \( t_i \) denote agent \( i \)'s consumption of the good. We call \( t_i \) agent \( i \)'s transfer: if \( t_i > 0 \), it is a transfer from the center to \( i \); if \( t_i < 0 \), \( |t_i| \) is a transfer from \( i \) to the center.

Agent \( i \)'s utility when she is assigned the set of tasks \( A_i \in 2^\mathbb{A} \) (note that \( A_i \) may be empty) and consumes \( t_i \in \mathbb{R} \) is

\[
u(A_i, t_i; c_i) = -c_i(A_i) + t_i.
\]

For each \( A \in 2^\mathbb{A} \) and each \( N \in \mathcal{N} \), let \( \mathcal{A}(A, N) = \{(A'_i)_{i \in N} : \text{for each } i \in N, A'_i \in 2^\mathbb{A} \text{, for each pair } \{i, j\} \subseteq N, A'_i \cap A'_j = \emptyset \text{, and } \bigcup_{i \in N} A'_i = A\} \).

For each \( N \in \mathcal{N} \), an assignment for \( N \) is a list \( (A_i)_{i \in N} \in \mathcal{A}(\mathbb{A}, N) \).

A transfer profile for \( N \) is a list \( (t_i)_{i \in N} \in \mathbb{R}^N \). An allocation for \( N \) is a list \( (A_i, t_i)_{i \in N} \) where \( (A_i)_{i \in N} \) is an assignment and \( (t_i)_{i \in N} \) is a transfer profile for \( N \).

A mechanism is a function \( \varphi \equiv (A, t) \) defined over the union \( \bigcup_{N \in \mathcal{N}} \mathcal{C}^N \) that associates with each economy an allocation: for each \( N \in \mathcal{N} \), each \( c \in \mathcal{C}^N \), and each \( i \in N \), \( \varphi_i(c) \equiv (A_i(c), t_i(c)) \in 2^\mathbb{A} \times \mathbb{R} \).

\(^3\)As usual, \( \mathbb{R}_+ \) denotes the set of non-negative real numbers.
For each \( N \in \mathcal{N} \), each \( c \in \mathcal{C}^N \), and each \( A \in 2^A \), let \( W(c, A) \) be the minimal total cost among all possible distributions of \( A \) to the agents in \( N \). That is,

\[
W(c, A) = \min \left\{ \sum_{i \in N} c_i(A'_i) : (A'_i)_{i \in N} \in \mathcal{A}(A, N) \right\}
\]

2.1 The Groves Mechanisms

Since the utility of each agent is increasing in her transfer, and her transfer and the total transfer can be of any size, every allocation is Pareto-dominated by some other allocation with higher transfers. That is, no allocation is Pareto-efficient. Still, we can define a notion of efficiency restricted to the assignment of the tasks. Since utilities are quasi-linear, given an economy, an allocation that minimizes the total cost incurred by all agents is Pareto-efficient for that economy among all allocations with the same, or smaller, total transfer. A mechanism is assignment-efficient if it chooses only such allocations.

**Assignment-Efficiency:** For each \( N \in \mathcal{N} \) and each \( c \in \mathcal{C}^N \),

\[
\sum_{i \in N} c_i(A_i(c)) = W(c, A_i).
\]

For each \( N \in \mathcal{N} \) and each \( c \in \mathcal{C}^N \), let \( A^*(c) \) be the set of efficient assignments for \( c \).

An assignment-efficient mechanism assigns the tasks so that the actual total cost is minimal if the agents report their true costs. Strategy-proofness requires that no agent ever benefits by misrepresenting her costs (Gibbard, 1973; Satterthwaite, 1975).

**Strategy-proofness**: For each \( N \in \mathcal{N} \), each \( i \in N \), each \( c \in \mathcal{C}^N \), and each \( c'_i \in \mathcal{C} \),

\[
u(\varphi_i(c); c_i) \geq \nu(\varphi_i(c'_i, c_{-i}); c_i).
\]

The so called Groves mechanisms were introduced by Vickrey (1961), Clarke (1971), and Groves (1973). A Groves mechanism chooses, for each economy, an efficient assignment of the tasks. In the literature, Groves mechanisms are sometimes defined as correspondences that select all the efficient assignments in an economy. We work with single-valued Groves mechanisms and assume that each Groves mechanism is associated with a tie-breaking rule that determines which of the efficient assignments (if there are more than one) is chosen. Let \( T \) be the set of all possible tie-breaking rules and \( \tau \) be a typical element of this set.

The transfer of each agent determined by a Groves mechanism has two parts. First, each agent pays the total cost incurred by all other agents at the assignment chosen by the mechanism. Second, each agent receives an amount of money that does not depend on her own cost. For each \( i \in N \), let \( h_i \) be a real-valued function defined over the union \( \bigcup_{N \in \mathcal{N}} \mathcal{C}^N \) such that for each \( N \in \mathcal{N} \) with \( i \in N \) and each \( c \in \mathcal{C}^N \), \( h_i \) depends only on \( c_{-i} \). Let \( h = (h_i)_{i \in N} \) and \( \mathcal{H} \) be the set of all such \( h \).

The Groves mechanism associated with \( h \in \mathcal{H} \) and \( \tau \in T \), \( G^{h, \tau} \): Let \( G^{h, \tau} \equiv (A^{\tau}, t^{h, \tau}) \) be such that for each \( N \in \mathcal{N} \), each \( i \in N \), and each \( c \in \mathcal{C}^N \),

\[
A^{\tau}(c) \in A^*(c)
\]

\footnote{See Thomson (2005) for an extensive survey on strategy-proofness.}
and
\[ t_i^{h,\tau}(c) = - \sum_{j \in N \setminus \{i\}} c_j(A_j^\tau(c)) + h_i(c_{-i}). \]

The following lemma will be of much use.

**Lemma 1.** For each \( N \in \mathcal{N} \), each \( i \in N \), and each \( c \in \mathcal{C}^N \),
\[ u(G_i^{h,\tau}(c); c_i) = -W(c, h) + h_i(c_{-i}). \]

By Lemma 1, for each \( h \in \mathcal{H} \), the mechanisms in \( \{G_i^{h,\tau}\}_{\tau \in T} \) are Pareto-indifferent.\(^5\) Hence, the particular tie-breaking rule used is irrelevant in the determination of the utilities.

The following theorem justifies our interest in Groves mechanisms.\(^6\)

**Theorem A** A mechanism is assignment-efficient and strategy-proof on \( \bigcup_{N \in \mathcal{N}} \mathcal{C}^N \) if and only if it is a Groves mechanism.

**Proof:** Since for each \( N \in \mathcal{N} \), \( \mathcal{C}^N \) is convex, the proof follows from Holmström (1979). \( \square \)

## 3 Egalitarian-Equivalence

If agents are equally responsible for the completion of the tasks, then assigning them identical bundles would be fair. But such an identical-bundle allocation may not always be feasible due to the heterogeneity in tasks. Fortunately, in our model, we can always find an allocation such that each agent is indifferent between her assigned bundle and a common reference bundle (consisting of a reference set of tasks and a reference transfer). Egalitarian-equivalence (Pazner and Schmeidler, 1978) requires that only such allocations be chosen.

**Egalitarian-equivalence:** For each \( N \in \mathcal{N} \) and each \( c \in \mathcal{C}^N \), there is a set of tasks (which may be empty) \( R \in 2^H \) and a transfer \( r \in \mathbb{R} \) such that for each \( i \in N \),
\[ u(\varphi_i(c); c_i) = u((R, r); c_i). \]

To characterize the class of egalitarian-equivalent Groves mechanisms, we need the following notation.

**Notation 1**

(a) For each \( i \in \mathbb{N} \), let \( c_i^0 \in \mathcal{C} \) be such that for each \( A \in 2^H \), \( c_i^0(A) = 0 \).

(b) For each \( N \in \mathcal{N} \), let \( \Pi_N \equiv \{ \pi|\pi: N \to \{1, 2, ..., n\} \text{ is a bijection} \} \).

For instance, if \( N = \{i, j\} \), then \( \Pi_N = \{ \pi, \pi' \} \) where \( \pi(i) = 1, \pi(j) = 2, \pi'(i) = 2 \), and \( \pi'(j) = 1 \).

(c) For each \( N \in \mathcal{N} \), each pair \( \{c, c'\} \subset \mathcal{C}^N \), each \( \pi \in \Pi_N \), and each \( j \in N \), let \( a(c, c', \pi, j) \in \mathcal{C}^N \) and \( b(c, c', \pi, j) \in \mathcal{C}^N \) be such that for each \( i \in N \),
\[ a(c, c', \pi, j)_i = \begin{cases} 
  c_i & \text{if } \pi(i) \leq \pi(j), \\
  c'_i & \text{if } \pi(i) > \pi(j). 
\end{cases} \]

---

\(^5\)Let \( N \in \mathcal{N} \) and \( c \in \mathcal{C}^N \). Allocations \( (A_i, t_i)_{i \in N} \) and \( (A'_i, t'_i)_{i \in N} \) are Pareto-indifferent for \( c \) if and only if for each \( i \in N \), \( u(A_i, t_i; c_i) = u(A'_i, t'_i; c_i) \). The mechanisms \( \varphi \) and \( \varphi' \) are Pareto-indifferent if for each \( N \in \mathcal{N} \), each \( i \in N \), and each \( c \in \mathcal{C}^N \), \( u(\varphi(c); c_i) = u(\varphi'(c); c_i) \).

---

5 Let \( N \in \mathcal{N} \) and \( c \in \mathcal{C}^N \). Allocations \( (A_i, t_i)_{i \in N} \) and \( (A'_i, t'_i)_{i \in N} \) are Pareto-indifferent for \( c \) if and only if for each \( i \in N \), \( u(A_i, t_i; c_i) = u(A'_i, t'_i; c_i) \). The mechanisms \( \varphi \) and \( \varphi' \) are Pareto-indifferent if for each \( N \in \mathcal{N} \), each \( i \in N \), and each \( c \in \mathcal{C}^N \), \( u(\varphi(c); c_i) = u(\varphi'(c); c_i) \).
Let us explain Notation 1(c). Let $N \in \mathcal{N}$, $\{c, c'\} \subset C^N$, and $\pi$ be a bijection from $N$ to $\{1, 2, \ldots, n\}$. Starting from $c'$, we can arrive at $c$ in $n$ steps: at each step $s$, we change the cost function of agent $j \in N$ with $\pi(j) = s$, from $c'_j$ to $c_j$. Hence, $\pi$ determines the order of agents in which their cost functions switch from the one in $c'$ to the one in $c$. The economy obtained at steps $\pi(j)$ and $\pi(j) - 1$ are denoted as $a(c, c', \pi, j)$ and $b(c, c', \pi, j)$, respectively. The only difference between $a(c, c', \pi, j)$ and $b(c, c', \pi, j)$ is that in the former, agent $j$'s cost function is $c_j$ and in the latter one, it is $c'_j$, that is, $a(c, c', \pi, j) = c_j$, $b(c, c', \pi, j) = c'_j$, and for each $i \in N \setminus \{j\}$, $a(c, c', \pi, j)_i = b(c, c', \pi, j)_i$.

For instance, if $N = \{1, 2, \ldots, n\}$ and for each $i \in N$, $\pi(i) = i$, then for each $j \in N$,

\[
\begin{align*}
a(c, c', \pi, j) &= (c_1, c_2, \ldots, c_{j-1}, c_j, c'_{j+1}, c'_{j+2}, \ldots, c_n), \\
b(c, c', \pi, j) &= (c_1, c_2, \ldots, c_{j-1}, c'_j, c'_{j+1}, c'_{j+2}, \ldots, c'_n).
\end{align*}
\]

See Example 1 in Appendix A for an illustration of how $a(c, c', \pi, j)$ and $b(c, c', \pi, j)$ are determined for a three-agent economy.

The following theorem characterizes the class of egalitarian-equivalent Groves mechanisms. Here, for each $N \in \mathcal{N}$, each pair $\{c^0, c\} \subset C^N$ where $c^0 = (c^0_i)_{i \in N}$, each $\pi \in \Pi_N$, and each $j \in N$, the function $R$ associates a reference set of tasks to the economy $a(c, c^0, \pi, j)$. For each $\{\pi, \pi'\} \subseteq \Pi_N$, the reference sets of tasks associated with economies $\{a(c, c^0, \pi, j)\}_{j \in N}$ and $\{a(c, c^0, \pi', j)\}_{j \in N}$ are related as in Equation (1). Equation (2) specifies the transfers of an egalitarian-equivalent Groves mechanism.

**Theorem 1. Egalitarian-Equivalence Theorem (EE-Theorem):**
A Groves mechanism $G_{h, \gamma}$ is egalitarian-equivalent if and only if there is a set-valued function $R: \mathbb{C} \to 2^h$ and a real-valued function $\gamma: \mathbb{N} \to \mathbb{R}$ satisfying the following conditions:

(i) for each $N \in \mathcal{N}$, each pair $\{c^0, c\} \subset C^N$, and each pair $\{\pi, \pi'\} \subseteq \Pi_N$,

\[
\sum_{j \in N} c_j(R(a(c, c^0, \pi, j))) = \sum_{j \in N} c_j(R(a(c, c^0, \pi', j))),
\]

(1)

(ii) for each $N \in \mathcal{N}$, each $i \in N$, each pair $\{c^0, c\} \subset C^N$, and each $\pi^i \in \Pi_N$ with $\pi^i(i) = n$,

\[
h_i(c^0) = \gamma(N) + \sum_{j \in N \setminus \{i\}} c_j(R(a(c, c^0, \pi^i, j))).
\]

(2)

A strengthening of egalitarian-equivalence is to require a particular set of tasks to be the reference set for all economies. Let $\overline{R} \in 2^h$.

**$\overline{R}$-Egalitarian-equivalence:** For each $N \in \mathcal{N}$ and each $c \in C^N$, there is $r \in \mathbb{R}$ such that for each $i \in N$,

\[
u(\varphi_i(c); c_i) = u((\overline{R}, r); c_i).
\]

Let $\overline{R} \in 2^h$. In the Egalitarian-Equivalence Theorem, if we impose $\overline{R}$-egalitarian-equivalence instead of egalitarian-equivalence, then for each $N \in \mathcal{N}$, each pair $\{c^0, c\} \subset C^N$, each $\pi \in \Pi_N$, and each $j \in N$, we have $R(a(c, c^0, \pi, j)) = \overline{R}$. Hence, we have the following characterization.
Corollary 1. For each \( R \in 2^\mathcal{N} \), a Groves mechanism \( G^{h,\tau} \) is \( R \)-egalitarian-equivalent if and only if there is a real-valued function \( \gamma : \mathcal{N} \to \mathbb{R} \) such that for each \( N \in \mathcal{N} \), each \( i \in N \), and each \( c \in C^N \),

\[
h_i(c_{-i}) = \gamma(N) + \sum_{j \in N \setminus \{i\}} c_j(R).
\]

By Corollary 1, a mechanism is assignment-efficient, strategy-proof, and \( \emptyset \)-egalitarian-equivalent if and only if the transfer function is of the following form: there is an equivalent mechanism that is Pivotal. Remember that a Groves mechanism is Pivotal if for each \( N \in \mathcal{N} \), each \( i \in N \), and each \( c \in C^N \), \( h_i(c_{-i}) = W(c_{-i}, A) \).

Corollary 2. On the class of two-agent economies, a Groves mechanism \( G^{h,\tau} \) is \( \emptyset \)-egalitarian-equivalent if and only if there is a real-valued function \( \gamma : \mathcal{N} \to \mathbb{R} \) such that for each \( N \in \mathcal{N} \), each \( i \in N \), and each \( c \in C^N \),

\[
h_i(c_{-i}) = W(c_{-i}, A) + \gamma(N).
\]

If a Groves mechanism is egalitarian-equivalent, then for each \( c \in \mathbb{C} \), it selects an allocation that is Pareto-indifferent to a reference bundle \( (R(c), r(c)) \). That is, there are functions \( R : \mathbb{C} \to 2^{\mathcal{N}} \) and \( r : \mathbb{C} \to \mathbb{R} \) which associate each economy with a reference set of tasks and a reference transfer, respectively. The functions \( R \) and \( r \) are not arbitrary. The next theorem presents a complete characterization of the reference-set-of-tasks function, \( R \), the reference-transfer function, \( r \), and the egalitarian-equivalent Groves mechanisms associated with \( R \) and \( r \). Note that equations (3) and (6) imply equations (1) and (2), respectively. Hence, the next theorem is an extension of EE-Theorem.

Theorem 2. Egalitarian-Equivalence Extended-Theorem (EEE-Theorem):
A Groves mechanism \( G^{h,\tau} \) is egalitarian-equivalent if and only if there is a reference-set-of-tasks function \( R : \mathbb{C} \to 2^{\mathcal{N}} \), a reference-transfer function \( r : \mathbb{C} \to \mathbb{R} \), and a real-valued function \( \gamma : \mathcal{N} \to \mathbb{R} \) satisfying the following conditions:

(i) for each \( N \in \mathcal{N} \), each pair \( \{c, c'\} \subset C^N \), and each pair \( \{\pi, \pi'\} \subset \Pi_N \),

\[
\sum_{j \in N} c'_j(R(b(c, c', \pi, j))) - \sum_{j \in N} c_j(R(a(c, c', \pi, j))) = \sum_{j \in N} c'_j(R(b(c, c', \pi', j))) - \sum_{j \in N} c_j(R(a(c, c', \pi', j))),
\]

(ii) for each \( N \in \mathcal{N} \), each \( i \in N \), each triple \( \{c^0, c, c'\} \subset C^N \) such that \( c_{-i} = c'_{-i} \), and each \( \pi \in \Pi_N \),

\[
\sum_{j \in N : \pi(j) \geq \pi(i)} c_j(R(a(c, c^0, \pi, j))) - c_i(R(c)) = \sum_{j \in N : \pi(j) \geq \pi(i)} c'_j(R(a(c', c^0, \pi, j))) - c'_i(R(c')) \]

In the literature, these mechanisms are also known by the following names: Vickrey mechanisms, Clarke mechanisms, and Second-price sealed-bid auctions.
(iii) for each $N \in \mathcal{N}$, each pair $\{c^0, c\} \subset \mathcal{C}^N$, each $i \in N$, and each $\pi \in \Pi_N$, 

$$h_i(c_{-i}) = \gamma(N) + \sum_{j \in N} c_j(R(a(c, c^0, \pi, j))) - c_i(R(c)),$$

(6)

4 Egalitarian-Equivalent Mechanisms with Bounded Deficits

If the center wants to use a mechanism that allocates the objects efficiently and induces the agents to report their true preferences, then it has to select a Groves mechanism. However, one often is also interested in the amount of budget deficit or surplus that is generated by the mechanism. It is well-known that no Groves mechanism balances the budget (Green and Laont, 1977). However, we can design Groves mechanisms that respect an upper bound on the deficit generated.

Suppose that for each population $N \in \mathcal{N}$, the center is willing to incur a deficit up to a certain amount $T(N)$ ($T(N)$ may be negative). Let $T : \mathcal{N} \rightarrow \mathbb{R}$. The following axiom requires that for each $N \in \mathcal{N}$, the deficit in any economy with population $N$, no matter what the costs of the agents in $N$, should never exceed $T(N)$.

$T$-Bounded-Deficit ($T$-BD): For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, 

$$\sum_{i \in N} t_i(c) \leq T(N).$$

The revenue (budget surplus) of the center is equal to the negative of the total transfer (budget deficit). Hence, if $G^{h, \tau}$ satisfies $T$-bounded-deficit, then for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $- \sum_{i \in N} t_i^{h, \tau}(c) \geq -T(N)$. That is, the center is guaranteed to generate a revenue at least as much as $-T(N)$. Note that such a guarantee is absent when a Pivotal mechanism is used.

In a related model to ours, Ohseto (2004) characterizes the class of $\emptyset$-egalitarian-equivalent Groves mechanisms that generate deficits that are bounded above by a real number. This number doesn’t depend on the cost functions. In a variable population setting, this "bounded-deficit" property corresponds to $T$-bounded-deficit. His model differs from the one we study in four important respects: in his model, the indivisible goods are identical (i.e., the cost of each task is same for an agent), the number of goods is strictly less than the number of agents, each agent can be assigned at most one indivisible good, and population is fixed.

Ohseto’s Theorem 3 (stated for the undesirable objects and variable-population setting) is the following:

A Groves mechanism $G^{h, \tau}$ satisfies $\emptyset$-egalitarian-equivalence and $T$-bounded deficit if and only if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $h_i(c_{-i}) \leq \frac{T(N)}{|N|}$.

We can show that not only his result still holds in our more general setting, but moreover, that even if we weaken the $\emptyset$-egalitarian-equivalence to egalitarian-equivalence, we still characterize the same class of Groves mechanisms as stated in our next theorem.\(^7\)

For each $\gamma : \mathcal{N} \rightarrow \mathbb{R}$, let $\mathcal{S}^\gamma \equiv \{G^{h, \tau}|$ for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $h_i(c_{-i}) = \gamma(N);$ and $\tau \in T\}$.\(^7\)

\(^7\)To prove this result, we need a completely different proof technique than Ohseto’s. Our result is not a simple extension of his result to our setting.
**Theorem 3.** Let $T : \mathcal{N} \to \mathbb{R}$. A Groves mechanism $G^{h,\tau}$ satisfies egalitarian-equivalence and $T$-bounded-deficit if and only if $G^{h,\tau} \in \mathcal{S}^{\tau}$ where $\gamma : \mathcal{N} \to \mathbb{R}$ is such that for each $N \in \mathcal{N}$,

$$\gamma(N) \leq \frac{T(N)}{|N|}.$$ (7)

By Corollary 1, an interesting implication of Theorem 3 is the following: assignment-efficiency, strategy-proofness, egalitarian-equivalence, and $T$-bounded-deficit together imply $\emptyset$-egalitarian-equivalence.

**Remark 1.** An egalitarian-equivalent Groves mechanism satisfies $T$-bounded-deficit only if it is $\emptyset$-egalitarian-equivalent.

It is easy to see that a Groves mechanism is $\emptyset$-egalitarian-equivalent if and only if it is egalitarian.

**Egalitarianism:** For each $N \in \mathcal{N}$, each pair $\{i, j\} \subseteq N$, and each $c \in \mathcal{C}^N$, $u(\varphi_i(c); c_i) = u(\varphi_j(c); c_j)$.

Egalitarianism may be seen as a desirable property in task assignments. One may argue that people should not be held responsible for their preferences since they are influenced by factors outside their control such as social conditioning or biological factors (Cohen, 1989; Roemer, 1998). In imposition problems, a further reason for people not be held responsible for their costs is that they do not have the right to refuse their task assignments. Hence, in each economy, all agents should experience the same utility. By Remark 1, we see that we attain egalitarianism if we require an egalitarian-equivalent Groves mechanism to generate bounded deficits.

Note that a mechanism generates no-deficit if it satisfies $T$-bounded deficit where $T$ is such that for each $N \in \mathcal{N}$, $T(N) = 0$. The following result follows from Theorem 3.

**Corollary 3.** A Groves mechanism $G^{h,\tau}$ satisfies egalitarian-equivalence and generates no-deficit if and only if $G^{h,\tau} \in \mathcal{S}^{\tau}$ where $\gamma : \mathcal{N} \to \mathbb{R}$ is such that for each $N \in \mathcal{N}$,

$$\gamma(N) \leq 0.$$ (8)

By Theorem 3, one can rank egalitarian-equivalent Groves mechanisms according to the maximal budget deficit (or the minimal budget surplus) they may generate. Hence, the center can select the mechanism that fits in its targets regarding the bounds on deficit or surplus.

Moreover, among all egalitarian-equivalent Groves mechanisms that respect a given upper bound on deficit, it is possible to specify the Groves mechanism (up to Pareto-indifference) that Pareto-dominates the others.8 This Pareto-dominant mechanism is also the one which has the minimal surplus.

**Corollary 4.** Let $\mathcal{S}^{T-\text{BD}} \equiv \{G^{h,\tau} \in \mathcal{S}^{\tau}\}$ for each $N \in \mathcal{N}$, $\gamma(N) = \frac{T(N)}{|N|}$; and $\tau \in T$ and $\mathcal{S}^{\text{ND}} \equiv \{G^{h,\tau} \in \mathcal{S}^{\tau}\}$ for each $N \in \mathcal{N}$, $\gamma(N) = 0$; and $\tau \in T$.

(i) Mechanisms in $\mathcal{S}^{T-\text{BD}}$ Pareto-dominate all Groves mechanisms that satisfy egalitarian-equivalence and $T$-bounded-deficit.

Mechanisms in $\mathcal{S}^{T-\text{BD}}$ have minimal surplus among all Groves mechanisms that satisfy egalitarian-equivalence and $T$-bounded-deficit.

---

8Let $N \in \mathcal{N}$ and $c \in \mathcal{C}^N$. Allocation $(A_i, t_i) \in N$ Pareto-dominates $(A'_i, t'_i) \in N$ for $c$ if and only if for each $i \in N$, $u(A_i, t_i; c_i) \geq u(A'_i, t'_i; c_i)$ with strict inequality for some $i \in N$. The mechanism $\varphi$ Pareto-dominates $\varphi'$ if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $u(\varphi(c); c_i) \geq u(\varphi'(c); c_i)$ and there are $N \in \mathcal{N}$, $c \in \mathcal{C}^N$, and $i \in N$ such that $u(\varphi(c); c_i) > u(\varphi'(c); c_i)$.
(ii) Mechanisms in $S^{ND}$ Pareto-dominates all Groves mechanisms that satisfy egalitarian-equivalence and generate no-deficit.

Mechanisms in $S^{ND}$ have minimal surplus among all Groves mechanisms that satisfy egalitarian-equivalence and generate no-deficit.

Besides the bounds on deficit, the center may also be interested in bounds on the welfare of the agents. The following welfare lower bound incorporates the notion that it is unfair for an agent, if the agent is assigned all the tasks but has to pay the center. Hence, the utility an agent would experience if she was assigned all of the tasks and zero transfer should be a lower-bound on her welfare.

The Stand-Alone Lower-Bound: For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $u(\varphi_i(c); c_i) \geq -c_i(A)$.

The following result follows from Corollary 3 and the Stand-alone Lower-Bound Proposition in Yengin (2008b).

**Corollary 5.** A Groves mechanism satisfies the stand-alone lower-bound and no-deficit if and only if it belongs to $S^{ND}$.

## 5 Other Fairness Axioms and Logical Relations

When assignment-efficiency and strategy-proofness are imposed, several logical relations hold between fairness axioms that do not exist otherwise. Before we state these logical relations, let us present some additional fairness axioms.

Perhaps, the most basic fairness notion is to require that whenever two agents have the same characteristics (e.g., the same cost functions), they should be treated equally.

**Equal Treatment of Equals:** For each $N \in \mathcal{N}$, each pair $\{i, j\} \subseteq N$, and each $c \in \mathcal{C}^N$ such that $c_i = c_j$, $u(\varphi_i(c); c_i) = u(\varphi_j(c); c_j)$.

The following axiom requires that the allocation is invariant with respect to the relabelling of agents in a given population.

**Anonymity:** For each $N \in \mathcal{N}$, each bijection $\pi : N \to N$, each $i \in N$, and each $c \in \mathcal{C}^N$, $\varphi_1(c) = \varphi_{\pi(i)}((c_{\pi(j)})_{j \in N})$.

Note that if a Groves mechanism $G^{h, \tau}$ is anonymous, then, given $N \in \mathcal{N}$, for each pair $\{i, j\} \subseteq N$, $h_i = h_j$. Obviously, anonymity implies equal treatment of equals. However, for Groves mechanisms, we show in Proposition 1 (i) that these two axioms actually characterize the same class.

Suppose that for each subset of tasks, agent $i$ incurs a cost that is at least as high as what agent $j$ incurs. If $j$ were assigned a lower utility than $i$, it would be as if $j$ were penalized for having lower costs. The following property is meant to prevent this situation.

For each $A \in 2^A$, if $c_i(A) \geq c_j(A)$, we write $c_i \geq c_j$.

**Order Preservation** \(^9\): For each $N \in \mathcal{N}$, each pair $\{i, j\} \subseteq N$, and each $c \in \mathcal{C}^N$ such that $c_i \geq c_j$, $u(\varphi_i(c); c_i) \leq u(\varphi_j(c); c_j)$.

\(^9\)The same axiom appears in Atlamaz and Yengin (2008). Also, a similar property appears in Porter, Shoham, and Tennenholtz (2004) under the name of “no-competence penalty”.
Another central fairness notion is that, each agent should find her bundle at least as desirable as the bundle of any other agent (Foley, 1967).

**No-envy:** For each $N \in \mathcal{N}$, each pair $\{i, j\} \subseteq N$, and each $c \in \mathcal{C}^N$,

$$u(\varphi_i(c); c_i) \geq u(\varphi_j(c); c_i).$$

Pápai (2003)\(^{10}\) proves that on the unrestricted domain, no Groves mechanism is envy-free. On the subadditive domain of cost profiles\(^{11}\), she characterizes the class of envy-free Groves mechanisms. It is easy to see that in general, no-envy implies equal treatment of equals. Observation 3 in Pápai (2003) states that all envy-free Groves mechanism are anonymous. We show further that under assignment-efficiency and strategy-proofness, no-envy implies order preservation, which in turn implies anonymity.

The following Proposition states the inclusion relations between several classes of Groves mechanisms.

**Proposition 1.** (i) A Groves mechanism satisfies equal treatment of equals if and only if it is anonymous.

(ii) If a Groves mechanism preserves order, then it is anonymous.

(iii) If a Groves mechanism is egalitarian-equivalent, then it preserves order.

(iv) If a Groves mechanism is envy-free, then it preserves order.

(v) If a Groves mechanism is $\emptyset$-egalitarian-equivalent, then on the additive and the subadditive domains, it is envy-free.

An alternative statement of Proposition 1 (ii) is as follows: assignment-efficiency, strategy-proofness, and order preservation together imply anonymity. Other parts of Proposition 1 can also be stated in a similar way.

6 Appendix A

**Example 1.** Let $N = \{1, 2, 3\}$ and $\{c, c'\} \subseteq \mathcal{C}^N$. For each $i \in N$, let $\pi(i) = i$. Let $\pi'(1) = 3$, $\pi'(2) = 1$, and $\pi'(3) = 2$. Then,

$$a(c, c', \pi, 1) = (c_1, c'_2, c'_3) = b(c, c', \pi, 1)$$
$$a(c, c', \pi, 2) = (c_1, c_2, c'_3) = b(c, c', \pi, 2)$$
$$a(c, c', \pi, 3) = c$$

and

$$a(c, c', \pi', 2) = (c'_1, c_2, c'_3) = b(c, c', \pi', 2)$$
$$a(c, c', \pi', 3) = (c'_1, c_2, c_3) = b(c, c', \pi', 3)$$
$$a(c, c', \pi', 1) = c$$

Now, we demonstrate the calculation of the equations in EEE-Theorem.

- By Condition (ia) of EEE-Theorem,

\(^{10}\)Pápai (2003) studies envy-free Groves mechanisms in the same model as ours. The only immaterial difference between her model and ours is that in her model, objects are desirable. Her results still hold if we adapt them to our costly objects setting.

\(^{11}\)If for each pair $\{A, A'\} \subseteq 2^k$, $c_i(A \cup A') \leq c_i(A) + c_i(A')$, then $c_i$ is subadditive.
Let Conditions (i), (ii), and (iii) of EEE-Theorem hold. Then, by (6), for each if and only if there is a reference-set-of-tasks function equivalent Egalitarian-Equivalence Lemma (EE-Lemma) A Groves mechanism First, we present the following result which is a direct implication of egalitarian-equivalence 7 Appendix B \[ \begin{align*}
[c'(R') - c_1(R(c_1, c_2', c_3'))] + [c'_2(R(c_1, c_2', c_3')) - c_2(R(c_1, c_2, c_3))] + [c'_3(R(c_1, c_2, c_3)) - c_3(R(c))] \\
= [c'_2(R(c') - c_2(R(c', c_2', c_3'))) + [c'_2(R(c_1', c_2', c_3')) - c_3(R(c_1', c_2, c_3))] + [c'_1(R(c_1', c_2, c_3)) - c_1(R(c))].
\end{align*} \]

For \( c' = c_0 \), the above equality reduces to
\[ c_1(R(c_1, c_2_0, c_3_0)) + c_2(R(c_1, c_2, c_3)) + c_3(R(c)) = c_2(R(c_0, c_2, c_3)) + c_3(R(c_0, c_2, c_3)) + c_1(R(c)). \]

Note that this equality satisfies Condition (i) of EE-Theorem.

- By Condition (ii) of EEE-Theorem, if \( \pi \) is used:
\[ r(c) = -W(c, \mathcal{A}) + c_1(R(c_1, c_2, c_3)) + c_2(R(c_1, c_2, c_3)) + c_3(R(c)) + r(c_0), \]
if \( \pi' \) is used:
\[ r(c) = -W(c, \mathcal{A}) + c_2(R(c_0, c_2, c_3)) + c_3(R(c_0, c_2, c_3)) + c_1(R(c)) + r(c_0). \]

- By Condition (iii) of EEE-Theorem, if \( \pi \) is used:
\[ h_1(c_2, c_3) = r(c_0) + [c_1(R(c_1, c_2, c_3)) + c_2(R(c_1, c_2, c_3)) + c_3(R(c))] - c_1(R(c)), \]
if \( \pi' \) is used:
\[ h_1(c_2, c_3) = r(c_0) + c_2(R(c_0, c_2, c_3)) + c_3(R(c_0, c_2, c_3)), \]

similarly,
\[ h_2(c_1, c_3) = r(c_0) + [c_1(R(c_1, c_2, c_3)) + c_2(R(c_1, c_2, c_3)) + c_3(R(c))] - c_2(R(c)), \]
\[ = r(c_0) + [c_2(R(c_0, c_2, c_3)) + c_3(R(c_0, c_2, c_3)) + c_1(R(c))] - c_2(R(c)), \]

and
\[ h_3(c_1, c_2) = r(c_0) + c_1(R(c_1, c_2, c_3)) + c_2(R(c_1, c_2, c_3)), \]
\[ = r(c_0) + [c_2(R(c_0, c_2, c_3)) + c_3(R(c_0, c_2, c_3)) + c_1(R(c))] - c_3(R(c)). \]

7 Appendix B

First, we present the following result which is a direct implication of egalitarian-equivalence and Lemma 1:

Egalitarian-Equivalence Lemma (EE-Lemma) A Groves mechanism \( G^{h, \pi} \) is egalitarian-equivalent if and only if there is a reference-set-of-tasks function \( R : C \rightarrow 2^A \) and a reference-transfer function \( r : C \rightarrow \mathbb{R} \) such that for each \( N \in \mathcal{N} \), each \( c \in C^N \), and each \( i \in N \),
\[ h_i(c_{-i}) = W(c, \mathcal{A}) + r(c) - c_i(R(c)). \]  \hfill (9)

Proof of Theorem 2 (Egalitarian-Equivalence Extended-Theorem):
Let Conditions (i), (ii), and (iii) of EEE-Theorem hold. Then, by (6), for each \( N \in \mathcal{N} \), each \( i \in N \), each \( c \in C^N \), and each \( \pi \in \Pi_N \),
\[ -W(c, \mathcal{A}) + h_i(c_{-i}) = -W(c, \mathcal{A}) + \sum_{j \in N} c_j(R(a(c, e_0, \pi, j))) + \gamma(N) - c_i(R(c)), \]
\[ = r(c) - c_i(R(c)), \]
where the last equality follows from (5) in EEE-Theorem. By the EE-Lemma, \( G^{h,\tau} \) is egalitarian-equivalent.

Conversely, let \( G^{h,\tau} \) be an egalitarian-equivalent Groves mechanism. Let \( N \in \mathcal{N} \) and \( \{c, c'\} \subset \mathcal{C}^N \). For each \( \pi \in \Pi_N \) and each \( j \in N \), let \( a^{\pi,j} \equiv a(c, c', \pi, j) \) and \( b^{\pi,j} \equiv b(c, c', \pi, j) \). Note that \( a^{\pi,j}_j = c_j \) and \( b^{\pi,j}_j = c'_j \). By the EE-Lemma, there are \( R : \mathbb{C} \to 2^A \) and \( r : \mathbb{C} \to \mathbb{R} \) such that for each \( \pi \in \Pi_N \) and each \( j \in N \),

\[
\begin{align*}
  h_j(a^{\pi,j}_j) &= W(a^{\pi,j}, \mathbb{A}) + r(a^{\pi,j}) - c_j(R(a^{\pi,j})) \quad \text{and} \\
  h_j(b^{\pi,j}_j) &= W(b^{\pi,j}, \mathbb{A}) + r(b^{\pi,j}) - c'_j(R(b^{\pi,j})).
\end{align*}
\]

(10) (11)

Since for each \( \pi \in \Pi_N \) and each \( j \in N \), \( a^{\pi,j}_j = b^{\pi,j}_j \), by (10) and (11),

\[
  r(a^{\pi,j}) = -W(a^{\pi,j}, \mathbb{A}) + W(b^{\pi,j}, \mathbb{A}) + r(b^{\pi,j}) + c_j(R(a^{\pi,j})) - c'_j(R(b^{\pi,j})).
\]

(12)

Note that for each \( j = n \) with \( \pi(j) = n \), \( a^{\pi,j} = c \) and for each \( j \in N \) with \( \pi(j) = 1 \), \( b^{\pi,j} = c' \). Also for each pair \( \{i, j\} \subset N \) such that \( \pi(i) = \pi(j) - 1 \), we have \( b^{\pi,j} = a^{\pi,i} \). Using these equalities and (12), by recursive substitution, we obtain

\[
  r(c) = -W(c, \mathbb{A}) + W(c', \mathbb{A}) + r(c') + \sum_{j \in N} c_j(R(a^{\pi,j})) - \sum_{j \in N} c'_j(R(b^{\pi,j})).
\]

(13)

As an illustration, let \( N = \{1, 2, 3\} \), \( \{c, c'\} \subset \mathcal{C}^N \), and \( \pi' \in \Pi_N \) be such that \( \pi'(1) = 3 \), \( \pi'(2) = 1 \), and \( \pi'(3) = 2 \) (see Example 1). For each \( j \in N \), (12) implies

\[
\begin{align*}
  r(a^{\pi',2}) &= -W(a^{\pi',2}, \mathbb{A}) + W(b^{\pi',2}, \mathbb{A}) + r(b^{\pi',2}) + c_j(R(a^{\pi',2})) - c'_j(R(b^{\pi',2})), \\
  r(a^{\pi',3}) &= -W(a^{\pi',3}, \mathbb{A}) + W(b^{\pi',3}, \mathbb{A}) + r(b^{\pi',3}) + c_j(R(a^{\pi',3})) - c'_j(R(b^{\pi',3})), \\
  r(a^{\pi',1}) &= -W(a^{\pi',1}, \mathbb{A}) + W(b^{\pi',1}, \mathbb{A}) + r(b^{\pi',1}) + c_j(R(a^{\pi',1})) - c'_j(R(b^{\pi',1})).
\end{align*}
\]

(14a) (14b) (14c)

As in Example 1, \( a^{\pi',1} = c \), \( b^{\pi',1} = a^{\pi',3} \), \( b^{\pi',3} = a^{\pi',2} \), and \( b^{\pi',2} = c' \). By using these equalities and solving (14a), (14b), and (14c) together, we obtain (13).

**Observation 1:** Note that (13) holds for each \( \pi \in \Pi_N \). Hence, for each pair \( \{\pi, \pi'\} \subset \Pi_N \),

\[
\begin{align*}
  \sum_{j \in N} c_j(R(a^{\pi,j})) - \sum_{j \in N} c'_j(R(b^{\pi,j})) &= r(c) + W(c, \mathbb{A}) - W(c', \mathbb{A}) - r(c'), \\
  &= \sum_{j \in N} c_j(R(a^{\pi,j})) - \sum_{j \in N} c'_j(R(b^{\pi,j})).
\end{align*}
\]

(15)

This completes the proof for Condition (i) of EEE-Theorem.

**Observation 2:** Let \( \gamma : \mathcal{N} \to \mathbb{R} \) be such that for each \( N \in \mathcal{N} \) and \( c^0 = (c^0_i)_{i \in N} \), \( \gamma(N) = r(c^0) \). Note that \( W(c^0, \mathbb{A}) = 0 \). Hence, if \( c' = c^0 \), by (13), for each \( \pi \in \Pi_N \),

\[
  r(c) = -W(c, \mathbb{A}) + \gamma(N) + \sum_{j \in N} c_j(R(a(c, c^0, \pi, j))).
\]

(16)

This completes the proof for Condition (ii) of EEE-Theorem.

**Observation 3:** The EE-Lemma and (16) together imply for each \( \pi \in \Pi_N \) and each \( i \in N \),

\[
  h_i(c_{-i}) = W(c, \mathbb{A}) + r(c) - c_i(R(c)) = \gamma(N) + \sum_{j \in N} c_j(R(a(c, c^0, \pi, j))) - c_i(R(c)).
\]

(17)
This completes the proof for Condition (iii) of EEE-Theorem.

**Observation 4:** Let $N \in \mathcal{N}$, $i \in N$, $\{c, c'\} \subseteq \mathcal{C}^N$ with $c_{-i} = c'_{-i}$, and $\pi \in \Pi_N$. Note that for each $j \in N$ with $\pi(j) < \pi(i)$, $a(c, c^0, \pi, j) = a(c', c^0, \pi, j)$. Since $c_{-i} = c'_{-i}$, then $h_i(c_{-i}) = h_i(c'_{-i})$. By (17), we obtain Condition (ib) of EEE-Theorem. ■

**Proof of Theorem 1 (Egalitarian-Equivalence Theorem):**

Let $N \in \mathcal{N}$ and $c \in \mathcal{C}^N$. Consider Observation 1 in the proof of EEE-Theorem. If $c' = c^0$, then by (15), for each pair $\{\pi, \pi'\} \subseteq \Pi_N$,

$$
\sum_{j \in N} c_j(R(a(c, c^0, \pi, j))) = \sum_{j \in N} c_j(R(a(c, c^0, \pi', j))).
$$

Hence, we obtain (1) in the EE-Theorem. Now, consider Observation 3 in the proof of EEE-Theorem. Let $i \in N$ and $\pi^i \in \Pi_N$ be such that $\pi^i(i) = n$. Note that $a(c, c^0, \pi^i, i) = c$. Thus, by (17),

$$
h_i(c_{-i}) = \gamma(N) + \sum_{j \in N} c_j(R(a(c, c^0, \pi^i, j))) - c_i(R(a(c, c^0, \pi^i, i))) = \gamma(N) + \sum_{j \in N \setminus \{i\}} c_j(R(a(c, c^0, \pi^i, j))).
$$

Hence, we obtain (2) in the EE-Theorem. ■

**Proof of Theorem 3:**

**“If” Part:** Let $G^{h, \gamma} \in \mathcal{S}^\gamma$ where $\gamma : \mathcal{N} \rightarrow \mathbb{R}$ is as in (7). By Corollary 1, it is $\emptyset$–egalitarian-equivalent. Note that for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$,

$$
\sum_{i \in N} t_i^{h, \gamma}(c) = -(n - 1)W(c, \mathbb{A}) + n\gamma(N).
$$

Let $T : \mathcal{N} \rightarrow \mathbb{R}$. By (7), $\sum_{i \in N} t_i^{h, \gamma}(c) \leq -(n - 1)W(c, \mathbb{A}) + T(N) \leq T(N)$. Hence, $G^{h, \gamma}$ satisfies $T$–bounded-deficit.

**“Only If” Part:** Let $T : \mathcal{N} \rightarrow \mathbb{R}$ and $G^{h, \gamma}$ be an egalitarian-equivalent Groves mechanism that satisfies $T$–bounded-deficit. The proof is in two parts. In Part 1, we show that $G^{h, \gamma}$ is $\emptyset$–egalitarian-equivalent. In Part 2, we show that $h$ is as in equality (7).

**Part 1:**

By $T$–bounded-deficit, for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$,

$$
\sum_{i \in N} t_i(c) = -(n - 1)W(c, \mathbb{A}) + \sum_{i \in N} h_i(c_{-i}) \leq T(N). \tag{18}
$$

Since $G^{h, \gamma}$ is egalitarian-equivalent, there are $R : \mathbb{C} \rightarrow 2^\mathbb{A}$ and $\gamma : \mathcal{N} \rightarrow \mathbb{R}$ such that (1) and (2) hold. By (2) and (18), for each $N \in \mathcal{N}$, each $i \in N$, each $\pi^i \in \Pi_N$, and each pair $\{c^0, c\} \subseteq \mathcal{C}^N$,

$$
-(n - 1)W(c, \mathbb{A}) + \sum_{i \in N} \sum_{j \in N \setminus \{i\}} c_j(R(a(c, c^0, \pi^i, j))) \leq T(N) - \varepsilon(N). \tag{19}
$$

If $c = c^0$, then the left-hand-side of inequality (19) is 0. Hence, for each $N \in \mathcal{N}$, there is $\varepsilon(N) \geq 0$ such that $T(N) - n\gamma(N) = \varepsilon(N)$.}

14
Without loss of generality, let \( N = \{1, 2, \ldots, n\} \subset N \), \( c^0 \in C^N \), and \( \varepsilon(N) = \varepsilon \). For each \( i \in N \), let \( \pi^i \in \Pi_N \) be such that for each \( j < i \), \( \pi^i(j) = j \), and for each \( j > i \), \( \pi^i(j) = j - 1 \). For instance, if \( N = \{1, 2, 3, 4, 5\} \) and \( i = 3 \), then \( \pi^i \) lists the agents in the following order: \( (1, 2, 4, 5, 3) \). For each \( c \in C^N \), let

\[
X(c) = \sum_{i \in N \setminus \{n\}} \sum_{j \in N \setminus \{i, n\}} c_j(R(a(c, c^0, \pi^i, j))), \\
Y(c) = \sum_{i \in N \setminus \{n\}} c_i(R(a(c, c^0, \pi^i, n))), \\
Z(c) = \sum_{j \in N \setminus \{n\}} c_j(R(a(c, c^0, \pi^n, j))).
\]

Then, (19) can be written as follows: for each \( c \in C^N \),

\[-(n - 1)W(c, \Lambda) + X(c) + Y(c) + Z(c) \leq \varepsilon.\]  

(20)

For each \( i \in N \) and each \( c \in C^N \), let \( c_i^* \in C \) be such that \( W(c_{-i}, \Lambda) = W((c_{-i}, c_i^*), \Lambda) \) and for each \( A \in (2^A \setminus \emptyset), c_i^*(A) > \varepsilon + (n - 1)W(c_{-i}, \Lambda) \geq \varepsilon \).

**Claim:** For each \( i \in N \), each \( j \in N \setminus \{i\} \), and each \( c \in C^N \),

(a) \( R((a(c, c^0, \pi^i, j_{-i}, c_i^*)) = \emptyset \) and

(b) \( c_j(R(a(c, c^0, \pi^i, j))) = 0 \).

**Proof of the claim:** We prove the claim for \( i = n \). The proof for the other agents is analogous.

For each \( j \in N \setminus \{n\} \) and each \( c \in C^N \), let \( c^j = (c_1, c_2, \ldots, c_j, c_{j+1}, \ldots, c_n) \) and \( c^*j = (c_1, c_2, \ldots, c_j, c_{j+1}^*, \ldots, c_n) \).

We need to show that for each \( j \in N \setminus \{n\} \) and each \( c \in C^N \), (a) \( R(c^j) = \emptyset \) and (b) \( c_j(R(c^j)) = 0 \).

Let \( c \in C^N \) and \( \tilde{\pi} \in \Pi_N \) be such that \( \tilde{\pi}(n) = 1 \) and for each \( j \in N \setminus \{n\}, \tilde{\pi}(j) = j + 1 \).

**Step 0:**

Let \( c^0 = (c_{1, c_2, \ldots, c_{n-1}, c_n}^*). \) Since for each \( j \in N \setminus \{n\} \) and each \( A \in 2^A, c^0_j(A) = 0 \), then \( X(c^0) = Z(c^0) = 0 \). Since for each \( i \in N \setminus \{n\}, a(c^0, c^0, \pi^i, n) = c^0 \), then \( Y(c^0) = (n - 1)c_n^*(R(c^0)) \). Note that \( W(c^0, \Lambda) = 0 \). Hence, by (20),

\[-(n - 1)c_n^*(R(c^0)) \leq \varepsilon.\]  

(21)

Since, for each \( A \in (2^A \setminus \emptyset), c_n^*(A) > \varepsilon \), for (21) to be true, we need \( R(c^0) = \emptyset \).

**Step 1:**

(a) Consider \( c^1 = (c_1, c_2, \ldots, c_n^0) \) and \( c^1* = (c_1, c_2^0, \ldots, c_{n-1}^0, c_n^*). \) Note that \( a(c^1, c^0, \pi^1, n) = c^0 \) and for each \( i \in N \setminus \{1, n\}, a(c^1, c^0, \pi^i, n) = c^1. \) Thus,

\[Y(c^1) = c_n^*(R(c^0)) + (n - 2)c_n^*(R(c^1)).\]  

(22)

By Step 0, \( R(c^0) = \emptyset \). This equality, the fact that \( W(c^1, \Lambda) = 0 \), and equations (20) and (22) together imply

\[X(c^1) + (n - 2)c_n^*(R(c^1)) + Z(c^1) \leq \varepsilon.\]  

(23)

Note that \( X(c^1) \geq 0, Z(c^1) \geq 0 \), and for each \( A \in (2^A \setminus \emptyset), c_n^*(A) > \varepsilon \). These inequalities and (23) together imply \( R(c^1) = \emptyset \).
b) Note that \( a(c^1, e^0, \pi^0, n) = c^1 \) and \( a(c^1, e^0, \tilde{\pi}, n) = c^0 \). Also, for each \( j \in N \setminus \{n\} \),
\( a(c^1, e^0, \pi^0, j) = c^1 \) and \( a(c^1, e^0, \tilde{\pi}, j) = c^1 \). These equalities and (1) together imply
\[
\sum_{j \in N} c_j^1(R(a(c^1, e^0, \pi^0, j))) = \sum_{j \in N} c_j^1(R(a(c^1, e^0, \tilde{\pi}, j))).
\]
\( c_1(R(c^1)) + c_n^*(R(c^1)) = c_1(R(c^1)) + c_n^*(R(c^0)). \) (24)

By Step 0, \( R(c^0) = \emptyset \) and by Step 1 (a), \( R(c^1) = \emptyset \). These equalities and (24) together imply \( c_1(R(c^1)) = 0 \).

**Induction Hypothesis:** Let \( k \in N \setminus \{n\} \). Suppose for each \( j \in \{1, 2, ..., k - 1\} \), \( R(c^j) = \emptyset \) and \( c_j(R(c^j)) = 0 \).

**Step k:**

a) Consider \( c^k \equiv (c_1, c_2, ..., c_k, c_{k+1}^0, ..., c_n^0) \) and \( c^{*k} \equiv (c_1, c_2, ..., c_k, c_{k+1}^0, ..., c_{n-1}^0, c_n^1) \).

Note the following:

(i) for each \( j \in \{1, 2, ..., k - 1\} \), \( a(c^{*k}, e^0, \pi^j, n) = (c_j^0, c_{j-1}^0) \),

(ii) \( a(c^{*k}, e^0, \pi^k, n) = c^{*k-1} \), and

(iii) for each \( j \in \{k + 1, ..., n - 1\} \), \( a(c^{*k}, e^0, \pi^j, n) = c^k \). By (i), (ii), and (iii),
\[
Y(c^{*k}) = \sum_{j \leq k-1} c_j^*(R((c_j^0, c_{j-1}^0))) + c_n^*(R(c^{*k-1})) + [n - k - 1]c_n^*(R(c^k)). \) (25)

By the induction hypothesis, \( R(c^{*k-1}) = \emptyset \). This equality and equations (20) and (25) together imply
\[
X(c^{*k}) + \sum_{j \leq k-1} c_j^*(R((c_j^0, c_{j-1}^0))) + [n - k - 1]c_n^*(R(c^{*k})) + Z(c^k) \leq \epsilon + (n - 1)W(c^k, \Delta). \) (26)

Note that \( (c_{n-1}, c_n^1) = c^{*n-1} \) and \( c_n^1 \) is such that \( W(c_{n-1}, \Delta) = W(c^{*n-1}, \Delta) \). Since for each \( A \in (2^\Delta \setminus \emptyset) \), \( c_n^1(A) > \epsilon + (n - 1)W(c_{n-1}, \Delta) \), \( W(c^{*n-1}, \Delta) \geq W(c^k, \Delta) \), \( X(c^k) \geq 0 \),
\[
\sum_{j \leq k-1} c_j^*(R((c_j^0, c_{j-1}^0))) \geq 0, \text{ and } Z(c^k) \geq 0, \text{ then inequality (26) is true if and only if } R(c^k) = \emptyset.

b) Note the following:

(i) for each \( j \in \{1, 2, ..., k\} \), \( a(c^{*k}, e^0, \pi^j, n) = c^j \) and \( a(c^{*k}, e^0, \tilde{\pi}, j) = c^{*j} \),

(ii) for each \( j \in \{k + 1, ..., n - 1\} \), \( a(c^{*k}, e^0, \pi^j, n) = c^k \) and \( a(c^{*k}, c^0, \tilde{\pi}, j) = c^{*k} \),

(iii) \( a(c^{*k}, e^0, \pi^j, n) = c^{*k} \) and \( a(c^{*k}, e^0, \tilde{\pi}, n) = c^{*0} \). By (i), (ii), (iii), and (1),
\[
\sum_{j \in N} c_j(R(a(c^{*k}, e^0, \pi^j, n))) = \sum_{j \in N} c_j(R(a(c^{*k}, e^0, \tilde{\pi}, j))).
\]
\[
\sum_{j \leq k-1} c_j(R(c^j)) + c_k(R(c^k)) + c_n^*(R(c^k)) = \sum_{j \leq k-1} c_j(R(c^j)) + c_k(R(c^k)) + c_n^*(R(c^0)). \) (27)

By the induction hypothesis, for each \( j \in \{1, 2, ..., k - 1\} \), \( R(c^j) = \emptyset \) and \( c_j(R(c^j)) = 0 \). By Step 0 and Step k (a), \( R(c^0) = R(c^k) = \emptyset \). These equalities and (27) together imply \( c_k(R(c^k)) = 0 \). This completes the proof of the claim.

By Claim (b), for each \( i \in N \) and each \( c \in C^N \),
\[
\sum_{j \in N \setminus \{i\}} c_j(R(a(c, e^0, \pi^i, j))) + c_i(R(a(c, e^0, \pi^i, i))) = c_i(R(a(c, e^0, \pi^i, i))) = c_i(R(c)). \) (28)
By (1), for each \( i \in N \), each \( l \in N \backslash \{i\} \), and each \( c \in C^N \),
\[
\sum_{j \in N} c_j(R(a(c, c^0, \pi^j, j))) = \sum_{j \in N} c_j(R(a(c, c^0, \pi^l, j))).
\]
This equality and (28) together imply for each \( i \in N \), each \( l \in N \backslash \{i\} \), and each \( c \in C^N \),
\[
c_i(R(c)) = c_l(R(c)).
\] (29)
For each \( c \in C^N \), let \( r(c) \in \mathbb{R} \) and \( R(c) \) together satisfy (9). Then, by (29), for each \( c \in C^N \), \( R'(c) = \emptyset \) satisfies (9) for \( r'(c) = r(c) - c_i(R(c)) \) for some \( i \in N \). That is, \( G^{h,\tau} \) is \( \emptyset \)-egalitarian-equivalent. This completes the first part of the proof.

\textbf{Part 2:}
Since \( G^{h,\tau} \) is \( \emptyset \)-egalitarian-equivalent, by (2), for each \( N \in \mathcal{N} \), each \( i \in N \), and each \( c \in C^N \),
\[
h_i(c_{-i}) = \gamma(N).
\]
This equality and (18) together imply for each \( N \in \mathcal{N} \) and each \( c \in C^N \),
\[
n\gamma(N) - T(N) \leq (n - 1)W(c, A).
\] (30)
That is, for each \( N \in \mathcal{N} \), \( n\gamma(N) - T(N) \leq \min_{c \in C^N} \{(n - 1)W(c, A)\} \). Since \( \min_{c \in C^N} W(c, A) = W(c^0, A) = 0 \), by (30), we have \( \gamma(N) \leq \frac{T(N)}{n} \). This completes the second part of the proof.

\textbf{Proof of Corollary 4:}
(i) Let \( G^{h,\tau} \) be a Groves mechanism that satisfies egalitarian-equivalence and \( T \)-bounded-deficit. By Theorem 3, \( G^{h,\tau} \) is such that for each \( N \in \mathcal{N} \), each \( i \in N \), and each \( c \in C^N \),
\[
h_i(c_{-i}) = \gamma(N) \leq \frac{T(N)}{|N|}.
\]
Then, \( u(G^{h,\tau}_i(c); c_i) = -W(c, A) + \gamma(N) \). Since utility is increasing in \( \gamma(N) \), \( G^{h,\tau} \) Pareto-dominates all Groves mechanisms that satisfy egalitarian-equivalence and \( T \)-bounded-deficit if and only if for each \( N \in \mathcal{N} \), \( \gamma(N) = \frac{T(N)}{|N|} \). That is, \( G^{h,\tau} \in S^{T-BD} \).

For each \( N \in \mathcal{N} \) and each \( c \in C^N \), the budget surplus is \(- \sum_{i \in N} l_i^{h,\tau}(c) = (n - 1)W(c, A) - \sum_{i \in N} h_i(c_{-i}) \). Hence, the surplus is minimal if and only if \( \sum_{i \in N} h_i(c_{-i}) \) is maximal. Note that \( \sum_{i \in N} h_i(c_{-i}) \) is maximal if and only if for each \( N \in \mathcal{N} \), \( \gamma(N) = \frac{T(N)}{|N|} \). Hence, \( G^{h,\tau} \) generates maximal surplus among all Groves mechanisms that satisfy egalitarian-equivalence and \( T \)-bounded-deficit if and only if \( G^{h,\tau} \in S^{T-BD} \).

(ii) The proof follows from part (i) and the fact that no-deficit is \( T \)-bounded-deficit where for each \( N \in \mathcal{N} \), \( T(N) = 0 \).

\textbf{Proof of Proposition 1:}
(i) It is easy to see that anonymity implies equal treatment of equals. Conversely, let \( G^{h,\tau} \) be Groves mechanism that satisfies equal treatment of equals. Let \( N \in \mathcal{N} \) and \( c \in C^N \). We will show that for each \( i \in N \) and each bijection \( \pi : N \rightarrow N \), \( h_i(c_{-i}) = h_{\pi(i)}(\pi(c)_{-\pi(i)}) \) where \( \pi(c) \equiv (c_{\pi(i)})_{i \in N} \).
Let \( \{i, j\} \subseteq N \) and \( \widehat{c} \in \mathcal{C}^N \) be such that \( \widehat{c}_i = c_j \) and \( \widehat{c}_{-i} = c_{-i} \). Since \( \widehat{c}_i = \widehat{c}_j \), by equal treatment of equals and Lemma 1, \( h_i(\widehat{c}_{-i}) = h_j(\widehat{c}_{-j}) \). This equality and the fact that \( \widehat{c}_{-i} = \widehat{c}_{-j} = c_{-i} \) together imply
\[
    h_i(c_{-i}) = h_j(c_{-i}).
\] (31)

Next, let \( \pi' : N \to N \) be a bijection such that \( \pi'(i) = j, \pi'(j) = i \), and for each \( l \in N\setminus\{i, j\} \), \( \pi'(l) = l \). Let \( \pi'(c) \equiv (c_{\pi'(l)})_{l \in N} \). Since \( \pi'(c)_{-\pi'(i)} = c_{-i} \), we have \( h_{\pi'(i)}(\pi'(c)_{-\pi'(i)}) = h_j(c_{-i}) \).

This equality and (31) together imply \( h_i(c_{-i}) = h_{\pi'(i)}(\pi'(c)_{-\pi'(i)}) \). Note that for any bijection \( \pi : N \to N \), starting from \( c \), we can obtain \( \pi(c) \) by carrying out of a finite sequence of pair-wise switching of labels of two agents one of whom is always in the \( i^{th} \) position. Hence, \( G^{h,\pi} \) is anonymous.

(ii) Let \( G^{h,\pi} \) be an order preserving Groves mechanism. Since order preservation implies equal treatment of equals, by Proposition 1 (i), \( G^{h,\pi} \) is anonymous.

(iii) Let \( G^{h,\pi} \) be an egalitarian-equivalent Groves mechanism. Let \( N \in \mathcal{N}, \{i, j\} \subseteq N \), and \( c \in \mathcal{C}^N \) be such that \( c_i \geq c_j \). Then, by Condition (iii) in EEE-Theorem, \( h_i(c_{-i}) \leq h_j(c_{-j}) \). By Lemma 1, \( G^{h,\pi} \) preserves order.

(iv) Let \( G^{h,\pi} \) be an envy-free Groves mechanism. Assume, by contradiction, that \( G^{h,\pi} \) does not preserve order. Then, there are \( N \in \mathcal{N}, \{i, j\} \subseteq N \), and \( c \in \mathcal{C}^N \) such that \( c_i \geq c_j \) and \( u(G^{h,\pi}_i(c); c_i) > u(G^{h,\pi}_j(c); c_j) \). By Lemma 1, \( h_i(c_{-i}) > h_j(c_{-j}) \). Then,
\[
    c_i(A_i^T(c)) - W(c, A) + h_i(c_{-i}) > c_j(A_j^T(c)) - W(c, A) + h_j(c_{-j}),
\]
\[
    -c_j(A_j^T(c)) - [W(c, A) - c_i(A_i^T(c))] + h_i(c_{-i}) > -W(c, A) + h_j(c_{-j}),
\]
\[
    -c_j(A_j^T(c)) + t_j^{h,\pi}(c) > -W(c, A) + h_j(c_{-j}).
\]

This inequality and Lemma 1 together imply \( u(G^{h,\pi}_i(c); c_i) > u(G^{h,\pi}_j(c); c_j) \), which contradicts no-envy.

(v) Let \( G^{h,\pi} \) be an \( \emptyset \)-egalitarian-equivalent Groves mechanism. Let \( \mathcal{C} \in \{\mathcal{C}_{ad}, \mathcal{C}_{sub}\} \). Assume, by contradiction, that \( G^{h,\pi} \) is not envy-free on \( \mathcal{C} \). Then, there are \( N \in \mathcal{N}, \{i, j\} \subseteq N \), and \( c \in \mathcal{C}^N \) such that \( u(G^{h,\pi}_i(c); c_i) < u(G^{h,\pi}_j(c); c_j) \). This inequality and Lemma 1 together imply
\[
    -W(c, A) + h_i(c_{-i}) < -c_j(A_j^T(c)) + t_j^{h,\pi}(c). \tag{32}
\]

By \( \emptyset \)-egalitarian-equivalence and the EE-Lemma, there is \( r \in \mathbb{R} \) such that for each \( l \in N \), \( h_l(c_{-l}) = W(c, A) + r \). Thus,
\[
    h_i(c_{-i}) = h_j(c_{-j}). \tag{33}
\]

By Lemma 1, \( -c_j(A_j^T(c)) + t_j^{h,\pi}(c) = -W(c, A) + h_j(c_{-j}) \). This equality, (32), and (33) together imply \( c_i(A_i^T(c)) < c_j(A_j^T(c)) \). Hence,
\[
    c_i(A_i^T(c)) + c_i(A_i^T(c)) < c_i(A_i^T(c)) + c_j(A_j^T(c)). \tag{34}
\]

Note that on the subadditive domain, \( c_i(A_i^T(c) \cup A_j^T(c)) \leq c_i(A_i^T(c)) + c_j(A_j^T(c)) \), which holds as an equality on the additive domain. This inequality and (34) together imply that it is less costly to assign both \( A_i^T(c) \) and \( A_j^T(c) \) to agent \( i \) rather than assigning these sets to \( i \) and \( j \), respectively. This contradicts that \( A^T(c) \) is an efficient assignment.
8 References


