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Abstract

We develop a model of favor exchange in a network setting where the cost of performing favors is stochastic. For any given favor exchange norm, we allow for the endogenous determination of the network structure via a link deletion game. We characterize the set of stable as well as equilibrium systems and show that these sets are identical. The most efficient network topology and favor exchange convention are generically shown to be not supported as equilibrium of the link deletion game. Our model provides a useful framework for understanding the topology of favor exchange networks. While the model exhibits positive externalities, its properties differ from the "information transmission" model *à la* Jackson and Wolinsky, as evidenced by the emergence of regular networks as opposed to star networks as stable and efficient network structures.

JEL Classification Codes: D85, C78, L14, Z13.

1 Introduction

Economists and social scientists have long been interested in relationships of favor exchange. Indeed, favor exchange forms an integral part of human

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interactions, be it in a local neighborhood, a workplace, an ethnic group, or even an online community.

In our social life, two salient features of favor exchanges prevail: First, exchanges are reciprocal; agents regularly find themselves both giving and receiving favors. Second, each community develops norms regarding when it is appropriate to turn down a favor request, and when it is not. These norms are enforced via the threat of discontinuation of the relationship, either bilaterally or multilaterally.

Until recently, most of the scholarly work on patterns of favor exchanges used pre-determined network structures as grids on which such interactions took place (see, e.g. Neilson [8]). However, a recent paper by Jackson et al [5] made the first attempt to simultaneously determine both the evolution of a network structure and the pattern of favor exchange on it. In this paper we further explore this agenda by providing a non-cooperative foundation of the network structure using a simultaneous move link-deletion game.

We model the informal exchange of favors in a network where the need for favors arises randomly and the value of a favor is exogenous, while the cost of providing a favor is stochastic. The community follows a social norm whereby when asked for a favor, an agent is expected to provide it as long as her cost is not too high. This social norm is denoted by a convention c^* such that for a cost below c^* one is expected to oblige to a favor request. The stochasticity of the cost is a novel feature of our model that offers a more realistic framework. In addition, various social norms with a more or less demanding obligation to perform favors can be captured through higher or lower conventions c^*. When in need of a favor, an agent approaches the least cost provider, among her neighbors, who then decides whether or not to perform the favor. Agents can detect violations of the convention and exclude from the network those who refuse to conform.[1] Such punishment captures the common sense idea that people take a dim view of a non-provider if she is seen to be refusing a favor in spite of having a reasonably low cost of obliging. Our aim is to co-determine a system, i.e., a network structure-convention pair for a given favor value, that supports the exchange of favors in the community.

Our analysis yields several results, such as the existence of multiple Par-

[1] We are implicitly assuming that the favor exchange network is embedded in an information exchange network such that each agent is able to obtain information about the compliance (or otherwise) of each of her neighbors dealings in their respective neighborhood. This is considerably weaker than the complete information assumption in Jackson et al.

ticipation Compatible, (restricted) Pairwise Stable and Nash Equilibrium systems. Participation Compatibility refers to the strictly positive expected utility agents derive from being part of the favor exchange community. Restricted Pairwise Stability is an adaptation to our link deletion game of the traditional notion of Pairwise Stability, whereby an agent has no incentives to delete a link. It is similar to the one employed by Jackson et al [5] and captures the idea that it takes time and coordination to form a new link while an existing link can be unilaterally severed. Finally, Nash Equilibrium systems are those resistant to multiple link deletions. Interestingly, we show that the set of Nash Equilibrium systems and the set of (restricted) Pairwise Stable systems are identical. This result, which had been previously proven for non-stochastic link maintenance cost (see Calvó-Armengol and Ilkilic [2]), thus applies to a more general environment. Furthermore, we identify efficient systems and show that the most efficient system is generically not stable. This result emanates from the fact that there is a positive externality (on to neighbors) from an agent having additional links, which is not internalized by the agent. Also, we show that, the favor efficient system requires its network to be complete. The rest of the paper is organized as follows: Section 2 presents the model and characterizes the expected utility of an agent; Section 3 describes Participation Compatible, (Restricted) Pairwise Stable and Nash Equilibrium systems, and their interrelationship; efficient systems and their stability are analyzed in Section 4; Section 5 discusses the results in context of the existing literature and concludes. All proofs and calculations are relegated to the Appendix.

2 The Model

2.1 General Framework

In this section, we describe the basic framework for studying favor exchange in a social network setting. Consider a finite set $\mathcal{N} = \{1, 2, ..., N\}$ (where $N \geq 3$) of agents connected in a social network represented by an undirected graph. A network g is a set of unordered pairs of agents $\{i, j\}$ denoting those agents which are joined.

The complete network g_{N-1} is the set of all possible subsets of $\mathcal{N}$ of size 2. The set of all possible networks on $\mathcal{N}$ is then $\{g | g \subseteq g_{N-1}\}$. We use $g \setminus \{i, j\}$ for the network g with link $\{i, j\}$ removed. We denote $\mathcal{N}_i(g) =$

$\{j \mid \{i,j\} \in g\}$ the set of agents who have a relationship with agent i in social network g. Agents in $\mathcal{N}_i(g)$ are agent i's neighbors, and $d_i(g) = |\mathcal{N}_i(g)|$ is the cardinality of the set $\mathcal{N}_i(g)$. Where possible, we use d_i and $\mathcal{N}_i$ instead of $d_i(g)$ and $\mathcal{N}_i(g)$ for brevity. We focus our analysis on non-empty connected networks. However, it can be extended to disconnected networks by analyzing each connected components separately.

Each agent in the network may need favors from, and perform favors for, her neighbors. In particular we assume that each agent has an equal probability $\frac{1}{N}$ of needing a favor. This contrasts with the basic framework of Jackson et al [5] where each pair has a given probability of being in a favor exchange relationship. In their set-up the probability that one needs a favor as well as that one receives a favor increases with the number of neighbors. Our set-up is different: an agent gets more requests for favors as the number of neighbors increases but he does not need more favors. The value of getting the favor is exogenous and $0 < v < \infty$. The cost c_i incurred by agent i from performing a favor is drawn from an iid, uniform distribution over $[0,1]$[2]. We assume that every agent can observe the realization of the costs of all her neighbors, and when in need of a favor, an agent asks her lowest cost neighbor to provide it.

When asked for a favor, there is an obligation to perform it provided the cost of performing it is not "too high"; this is the social norm governing the favor exchange in the community. In particular, there exists a cost threshold c^* such that an agent i is supposed to perform (not perform) the favor when $c_i \leq c^*$ ($c_i > c^*$). Adherence to this convention is enforced by the threat of expulsion from the network in the event of non-compliance. The convention c^*, which applies to all agents in the network, captures the tolerance level of the population; a society with a lower c^* is more tolerant of non-performance of favors.

Positing the existence of a favor exchange norm is an innovation of our paper. This approach, in contrast to the existing literature on network formation, endogenizes the benefits and, in particular, the costs of maintaining a link.

[2]Our results hold for more general distributions; however, the uniform distribution enables us to characterize closed form solutions.

2.2 Expected Utility

The expected payoff agent i anticipates, from being in network g where convention c^* prevails and the value of a favour is v, is denoted $EU^i(g, c^*; v)$. It is composed of two elements: the gain from the favors her neighbors can potentially perform for her; the expected cost incurred by performing favors for some neighbors. The social welfare is then obtained by summing up the expected utilities across all individuals. Formally, social welfare can be written as

$$\frac{1}{N} v \cdot \sum_i F_i(c^*) - \frac{1}{N} \sum_i \{ \sum_{j \in \mathcal{N}_i} F_{ji}(c^*) \cdot E_j(c^*) \} \tag{1}$$

where $E_j(c^*)$ stands for $E(c_j | c_j \leq c^*)]$, i.e., the expected cost of a favor done for j, F_{ji} denotes the probability that i provides the favor needed by j and F_i denotes the probability that i received a favor when needed. By rearranging the summation we can express the same formula in terms of the sum of the individual benefits from a favor minus the social cost, borne by the neighbors, of performing the favor, i.e., the social welfare can be written as

$$\frac{1}{N} \sum_{i=1}^{N} [v - E_i(c^*)] \cdot F_i(c^*) \tag{2}$$

While most of our qualitative results hold for more general cost distributions, we will be using the uniform distribution for costs to provide closed form solutions. Hence, it is useful to establish the following claim.

Claim 1 *When costs are i.i.d, according to a uniform distribution on* $[0, 1]$ *then the expected utility of an individual* i *under a convention* c^* *is given by*

$$\begin{aligned} EU^i(g, c^*; v) &= v \cdot [1 - (1 - c^*)^{d_i}] \\ &- \sum_{j \in \mathcal{N}_i(g)} \frac{1}{d_j} \left\{ \frac{1}{d_j + 1} - c^*(1 - c^*)^{d_j} - \frac{1}{d_j + 1}(1 - c^*)^{d_j + 1} \right\} \end{aligned} \tag{3}$$

Proof: See Appendix.

Changes in agent i's degree may or may not be beneficial for her; it depends on the value of c^* and v. However, we can show that an increase in the degree of at least one of agent i's neighbors provides her with an increased level of expected utility.

Lemma 1 *Let $g' = g \cup \{j, k\}$ where $k \neq i$ and $j \in \mathcal{N}_i(g)$. Then, given c^* and v, we have $EU^i(g', c^*; v) \geq EU^i(g, c^*; v)$.*

Proof: See Appendix.

To understand the intuition behind Lemma 1, consider agent j who is agent i's neighbor. The more neighbors agent j has, the more likely it is that she will approach someone other than agent i for a favor. Hence, agent j having more neighbors decreases agent i's expected cost while keeping her expected benefits unchanged. As we will discuss later, this feature differentiates our model from the "information exchange networks" and "co-author networks" examined in Jackson and Wolinski [4]. In the former, there is always a benefit from adding more indirect links of any order n (not just $n = 2$), while in the latter, there is a crowding out effect of having second order indirect links.

3 Participation, Pairwise Stability and Equilibrium

3.1 Participation Compatibility

We assume that non participating agents, i.e., those who decide to stay out of the network, get a utility of zero. Therefore, an agent participates in the network if and only if she expects positive utility from it. As a tie-breaking convention, agents with zero expected utility, although indifferent, are assumed to stay out of the network. We refer to the condition $EU^i(g, c^*; v) > 0$ for all $i \in \mathcal{N}$ as the Participation Compatibility condition.

Definition 1 (Participation Compatibility) *A system $(g, c^*; v)$ is said to be Participation Compatible if $EU^i(g, c^*; v) > 0$, for all $i \in \mathcal{N}$.*

To identify whether such systems exist, we first restrict our attention to star networks $g_{S_{i,N-1}}$. In that case, agent i, the centre of the star, has a link with every other agent in the network, while her neighbors have one link each. Technically, this means that

$$\mathcal{N}_i(g_{S_{i,N-1}}) = \{1,\ 2, ..., i-1, i+1..., N\}$$

and

$$N_j(g_{S_{i,N-1}}) = \{i\} \text{ for all } j \in \mathcal{N} \setminus \{i\}\,.$$

We therefore have

$$EU^i(g_{S_{i,N-1}}, c^*; v) = v[1 - (1 - c^*)^{N-1}] - \frac{N-1}{2}(c^*)^2 \tag{4}$$

The next lemma states that there always exists a (non-zero) convention such that the star agent in a star network gets strictly positive expected utility. This holds for any value v.

Lemma 2 *Given v, there exist $c_i^* \in (0, 1]$ such that,*

$$EU^i\left(g_{S_i,N-1}, c_i^*; v\right) > 0.$$

Proof: See Appendix.

Our interest in star networks is intentional: given a network g with N agents, we can focus on the N star networks formed by each agent $i \in \mathcal{N}$ and their neighborhood. Applying Lemma 2, we then find c^* such that all N systems are Participation Compatible. Finally, by invoking Lemma 1, we establish the following Theorem.

Theorem 1 *For any value v and any network g, there exist $c^* \in (0, 1]$ such that $(g, c^*; v)$ is Participation Compatible.*

Proof: See Appendix.

While Theorem 1 is presented in terms of finding a convention c^* for a given network g, the reader may, depending on the context, find it more natural to think of finding a network g for a given social norm c^* such that the system $g, c^*; v$ is Participation Compatible. This is the dual problem. We will later revisit this problem and show there exists an upper bound such that for any c^* below it, one can find a network g that is Participation Compatible. Therefore, one can always modify a system, either by changing g or c^*, to make it Participation Compatible. However, whether or not such system can be maintained relies on whether any agent has any incentives to modify the structure of her neighborhood.

3.2 Restricted Pairwise Stability

For a given system $(g, c^*; v)$ to remain unchanged, it has to be that no agent is better off by severing links with any of her neighbors. This is part of the condition of Pairwise Stability. However, since we focus exclusively on link deletion, we need to adapt the requirements of Pairwise Stability to our environment; we hence consider Restricted Pairwise Stability.

Definition 2 (Restricted Pairwise Stability) *A system $(g, c^*; v)$ is said to be Restricted Pairwise Stable if, for all $i \in \mathcal{N}$ and all $j \in \mathcal{N}_i(g)$:*

$$EU^i(g, c^*; v) \geq EU^i(g \setminus \{ij\}, c^*; v) \;\; \textit{and} \;\; EU^j(g) \geq EU^j(g \setminus \{ij\}, c^*; v)$$

Similarly to the Participation Compatibility analysis, the problem can be interpreted as identifying a convention such that the system $(g, c^*; v)$ is Restricted Pairwise Stable (RPS).

Theorem 2 *Given any value v and network g, there exists c^*_{RPS} such that $(g, c^*; v)$ is Restricted Pairwise Stable for any $c^* \in [0, c^*_{RPS}]$.*

Proof: See Appendix.

We will later on show the dual problem, of finding a network g for a given convention c^* such that the system $(g, c^*; v)$ is RPS, can be solved for any convention below an upper bound.

Given v and g, the RPS condition between agent i and agent $j \in \mathcal{N}_i(g)$, which states that agent i prefers to keep her link with agent j rather than deleting it, can be written as:

$$vc^*_{ij}(1 - c^*_{ij})^{d_i - 1} \geq \frac{1}{d_j} \left\{ \frac{1}{d_j + 1} - c^*_{ij}(1 - c^*_{ij})^{d_j} - \frac{1}{d_j + 1}(1 - c^*_{ij})^{d_j + 1} \right\} \quad (5)$$

Denote $\tilde{c}^*_i = \min_{j \in \mathcal{N}_i} \left\{ \tilde{c}^*_{ij} \right\}$, where $\tilde{c}^*_{ij}$ is the highest c^*_{ij} for which Equation 5 holds. It can be shown that the RPS condition holds between agent i and any agent $j \in \mathcal{N}_i$ when $c^*_i \in [0, \tilde{c}^*_i]$. Thus, for a system $(g, c^*; v)$ to be RPS, it has to be that $c^* \leq c^*_{RPS}$, where $c^*_{RPS} = \min_{i \in \mathcal{N}} \{\tilde{c}^*_i\}$. In effect, it means that the most tolerant agent, i.e. the one requiring the lowest convention, is also the one who imposes her preferences on others. We refer to this agent as the Dictator of Tolerance. Formally,

Definition 3 (Dictator of Tolerance) *Given a system* $(g, c^*; v)$*, if* $\tilde{c}_i^* = c_{RPS}^*$*, then agent* i *is called The Dictator of Tolerance.*

Obviously, there always exists a Dictator of Tolerance. Hence, given a network g and value v, it is always possible to characterize a RPS system by adopting a convention for which the Dictator of Tolerance has no incentives to delete links.

We now investigate whether a RPS system is also Participation Compatible. It turns out that RPS requirements suffice to ensure Participation Compatibility, as stated in Proposition 1.

Proposition 1 *If the system* $(g, c^*; v)$ *where* $c^* \in (0, c_{RPS}^*]$ *is Restricted Pairwise Stable, then it is Participation Compatible.*

Proof: See Appendix.

The Participation Compatible condition guarantees positive expected utility from the favor exchange system. At the same time, the RPS one insures that no agent desires to sever links with her neighbors. This leads to the formal definition of a strategy in our environment, and to the characterization of the conditions required for such RPS systems to be in equilibrium.

3.3 Equilibrium

We study a game similar to the network formation literature.[3] However, our model differs in one important aspect: agents are initially endowed with a network g where convention c^* prevails and then choose which links they would like to *keep*. Our game is not one of network formation, but one of link deletion.

Starting with initial network g, each agent $i \in \mathcal{N}$ simultaneously announces which of the links $\{ij\}$ for all $j \in \mathcal{N}_i(g)$ she intends to keep. $S_i = \{0, 1\}^{N-1}$ is agent i's set of pure strategies. The generic strategy $s_i \in S_i$ is an $(N-1)$-tuple, such that $s_i = (s_{i1}, ..., s_{i,i-1}, s_{i,i+1} ..., s_{iN})$.

If agent i opts to keep her link with agent j, then s_{ij} is one; otherwise, it is zero. Because agents can only delete links, we impose that if $\{ij\}$ is not in the initial network g, then $s_{ij} = s_{ji} = 0$. As is standard, mutual

[3] For instance: Jackson [4], Goyal and Joshi [3], Calvó-Armengol [2]. Also see Jackson 2004 for an excellent survey.

consent is necessary for maintaining links, i.e. $\{ij\}$ is maintained if and only if $s_{ij} = s_{ji} = 1$.

Let $S = S_1 \times S_2 \times ... \times S_N$. A strategy profile $s = (s_1, s_2, ..., s_n) \in S$ thus induces a network $g(s)$. The mapping $S \mapsto g_N$ is not one-to-one since $s_{ij} \neq s_{ji}$ and $s_{ij} = s_{ji} = 0$ both result in $\{ij\}$ being absent from the network.

Before considering the system in its entirety, we first focus on the meaning and consequences in network terms from considering a Nash Equilibrium strategy profile.

Definition 4 (Nash Equilibrium Network) *A network g is a Nash Equilibrium Network if there exists a strategy profile s which constitutes a Nash Equilibrium strategy profile for the normal form link deletion game with initial network g and*

$$g = g(s)$$

where $g(s)$ is the network induced by s.

Since a link can only be maintained through mutual agreement, a Nash Equilibrium network can only be mapped back to a unique strategy profile, as stated by the following Lemma.

Lemma 3 *Any Nash Equilibrium Network admits a unique strategy profile.*

Now that the potential complications induced by a mapping that is not one-to-one have been addressed, we present our definition of Nash Equilibrium Systems.

Definition 5 (Nash Equilibrium System) *A system $(g, c^*; v)$ is said to be a Nash Equilibrium System if its network g is a Nash Equilibrium Network.*

We began our analysis by characterizing RPS systems of favor exchange on a network. We then introduced the noncooperative notion of a Nash equilibrium of a link deletion game. Our next result shows that the two concepts, coming while emanating from different motivations, are closely linked.

Theorem 3 *A system $(g, c^*; v)$ is Restricted Pairwise Stable if and only if it is a Nash Equilibrium System.*

Proof: See Appendix.

The "if" part of the proof is trivial. The "only if" part of the proof involves showing that a Nash Equilibrium System is characterized by a one-shot deviation property, i.e., if deleting one link is not profitable, then deleting any subset of links is not profitable either.

The absence of closed-form solutions restricts our analysis to the existence, rather than the characterization, of systems that fulfill all three requirements: Participation Compatible, Restricted Pairwise Stable and Nash Equilibrium System. However, in the next section, we address the question of efficiency of such systems and provide, for particular network topologies, very prescriptive results.

4 Efficient Systems

4.1 General Results

In this section, we focus on the efficiency of a system in the traditional sense. Given some network g and value v, we investigate whether there exists a convention c^* such that $\Sigma_{i=1}^N EU^i(g, c^*; v)$ is maximized. We find that such systems exist, although they may not be equilibria.

Definition 6 (Convention Efficiency) *A system* $(g, c^*; v)$ *is said to be Convention Efficient if for all* $c^{*'}$

$$\Sigma_{i=1}^N EU^i(g, c^*; v) \geq \Sigma_{i=1}^N EU^i(g, c^{*'}; v).$$

Our first efficiency result is intuitive: it states that for given a network g, a Convention Efficient system is found by adopting the convention that equates the marginal cost of providing a favor with its marginal benefit (subject to the boundary conditions).

Proposition 2 *Given a network* g *and value* v*, the system* $(g, c^*; v)$ *is Convention Efficient if and only if* $c^* = \min\{v, 1\}$.

Proof: See Appendix.

Although attractive, Convention Efficient systems are only desirable if they can be maintained, i.e., if they are Nash Equilibria. A Nash Equilibrium

System can arise only if the value v of performing a favor is below c^*_{RPS}. Recall that c^*_{RPS} is the highest convention that the Dictator of Tolerance is willing to follow without deleting links.

Theorem 4 *Given a network g, if $v > c^*_{RPS}$, then no Nash Equilibrium System $(g, c^*; v)$ is Convention Efficient.*

Proof: This follows directly from the proof of Theorem 2 where it is shown that $c^*_{RPS} < 1$, and Proposition 2. □

In the next subsection we provide complete characterizations of Nash Equilibrium Systems and their efficiency properties for particular network topologies that commonly arise in the network literature.

4.2 Specific Network Topologies

In what follows, we take a close look at star and regular networks. For star networks, we find the sufficient condition on v that ensures the system $(g_{S_{i,N}}, c^*; v)$ is a Convention Efficient Nash equilibrium. We present this result in the following Proposition:

Proposition 3 *The Convention Efficient system $(g_{S_{i,N-1}}, c^*; v)$ is a Nash Equilibrium System if $v \leq 1 - (\frac{1}{2})^{\frac{1}{N-2}}$.*

Proof: See Appendix.

For regular networks, we also identify the sufficient condition on v for which a Convention Efficient system is an Nash equilibrium. A network is said to be a regular network of degree n if each agent has exactly n links.

Proposition 4 *Consider g_n, the regular network of degree n.*
The Convention Efficient system $(g_n, c^*; v)$ is a Nash Equilibrium System if $v \leq 1 - \left(\frac{n-1}{2n}\right)^{\frac{1}{n+1}}$.

Proof: See Appendix.

Proposition 4 allows us to present the dual results corresponding to Theorem 1 and Theorem 2. Given any $c^* \leq v \leq 1 - \left(\frac{n-1}{2n}\right)^{\frac{1}{n+1}}$, we can find a network g, namely the regular network of degree n, that will satisfy the Participation and Restricted Pairwise Stability constraints.

Although this paper takes the initial network as given, it is natural to question whether a particular topology can provide consistently a higher sum of expected utilities.

Definition 7 (Network Efficiency) *Given v and c^*, a system $(g, c^*; v)$ is said to be Network Efficient if*

$$\Sigma_{i=1}^{N} EU^i(g, c^*; v) \geq \Sigma_{i=1}^{N} EU^i(g', c^*; v).$$

Not surprisingly, the complete network, g_{N-1}, where each agent has links with all other agents in the system, presents such characteristic.

Lemma 4 *For all $c^* \leq v$, the system $(g_{N-1}, c^*; v)$ is a Network Efficient system.*

Proof: See Appendix.

However, restrictions on v are required in order for $(g_{N-1}, c^*; v)$ to also be a Nash Equilibrium System.

Proposition 5 *If $v \leq 1 - \left(\frac{N-2}{2(N-1)}\right)^{\frac{1}{N}}$ and $c^* = v$, then $(g_{N-1}, c^*; v)$ is both, a Convention Efficient and a Network Efficient Nash Equilibrium System.*

Proof: Follows directly from Lemma 4 and Proposition 4.

Efficiency may not be the only desirable feature for these systems and in what follows, we explore possible avenues of interest, such as the kind of topology that is needed to create a community where performance of favors is expected.

4.3 Favor Efficiency

The notion of efficiency we used so far corresponded to the standard utilitarian notion of maximizing the sum of expected utilities across agents. An alternative definition of efficiency may be provided based on the probability with which an efficient favor gets performed and an inefficient favor does not get performed. A favor is said to be efficient if $v \geq c^*$. Similarly, we refer to an inefficient favor as one that is performed when $v < c^*$.

Denote $c_{\mathcal{N}_i}$ the random variable $\min\{c_j\}_{j \in \mathcal{N}_i}$. We refer to $\delta(g, c^*; v)$ [resp. $\varepsilon(g, c^*; v)$] as the probability that an Efficient [resp. Inefficient] Favor gets performed in a system $(g, c^*; v)$. Likewise, we define δ_i [resp. ε_i] as the probability of an efficient [resp. inefficient] favor getting performed *for* agent i. It follows that:

$$\delta = \frac{1}{N} \sum_{i=1}^{N} \delta_i \quad \text{and} \quad \varepsilon = \frac{1}{N} \sum_{i=1}^{N} \varepsilon_i$$

where

$$\delta_i = Pr(\text{Efficient Favor}) \cdot Pr(\text{The favor gets performed}|\text{Efficient Favor})$$

i.e.,

$$\delta_i = Pr(c_{\mathcal{N}_i} \leq v) \cdot Pr(c_{\mathcal{N}_i} \leq c^* | c_{\mathcal{N}_i} \leq v),$$

i.e.,

$$\delta_i = Pr(c_{\mathcal{N}_i} \leq \min\{c^*, v\}),$$

and

$$\varepsilon_i = Pr(c_{\mathcal{N}_i} \in (v, c^*]).$$

A measure of favor efficiency may then be defined as a function $W(\delta(g, c^*; v), \varepsilon(g, c^{*\prime}; v))$ such that $W_1 > 0$ and $W_2 < 0$. For example, $W = \delta(1 - \varepsilon)$.

Definition 8 (Favor Efficient Convention) *Given a network g and value v, a convention c^* is said to be Favor Efficient if for all $c^{*\prime}$*

$$W(\delta(g, c^*; v), \varepsilon(g, c^*; v)) \geq W(\delta(g, c^{*\prime}; v), \varepsilon(g, c^{*\prime}; v))$$

Proposition 6 *Given a network g and value v, a convention c^* is favor efficient if and only if $c^* = v$.*

Proof: Compared to any convention $c^* < v$, an alternative convention $c^{*\prime} \in (c^*, v)$ yields an equivalent similar ε but a bigger δ, so c^* could not have been efficient. Likewise, for any $c^* > v$, picking an alternative $c^{*'} \in (v, c^*)$ yields a lower ε but an equivalent δ. □

We can suitably modify the definition to consider *restricted* favor efficient convention by requiring that c^* must maximize the value of W over a given set of conventions $\mathcal{C}$, say set of RPS conventions.

Definition 9 (Favor Efficient Network) *For a given convention c^* and value v, a network g is said to be Favor Efficient if for all g'*

$$W(\delta(g, c^*; v), \varepsilon(g, c^*; v)) \geq W(\delta(g', c^*; v), \varepsilon(g', c^*; v)).$$

Proposition 7 *For any $0 < c^* \leq v$, network g is Favor Efficient if and only if it is the complete network g_{N-1}.*

Proof: See Appendix.

5 Discussion and Conclusion

In this paper we developed a model to study co-evolution of favor exchange norm and the network structure of favor exchange. We incorporated two features of real world into our model: First, the cost of providing favors is stochastic. Second, similar societies can adopt different conventions as to when it is acceptable for an agent to not perform a favor asked by another. We characterized the set of restricted stable networks and studies how they vary with the underlying favor-exchange convention. We provided the micro-foundations for the network formation process by describing the Nash equilibria if a link deletion game, which we showed were identical to the set of restricted stable networks. Our analysis suggested that the most efficient stable network are regular and the their degree is lower (higher) if the favor exchange convention is more (less) demanding.

We are interested in understanding what community characteristics, in terms of its network structure g, are conducive to having a more or less demanding favor exchange norm. Similarly, given an existing convention c^*, we would like to understand the network structures that are sustainable. We do not take a position as to the primacy of g over c^*, or vice versa; indeed either one may be treated as a primal, depending on the specific application.

The structure of our model gives rise to positive externalities from links, similar to the models by Jackson and Wolinsky [4], Johnson and Giles [7], Calvó-Armengol [1] and Jackson and Rogers [6]. However, unlike information transmission models, our externality comes form a different source: it comes from the fact that an agent having more friends reduces her dependence on the existing set of friends for favors. This leads to a structure were a person with many existing friends is an attractive friend to have. At the same time, such a person will not be interested in adding/keeping friends unless they themselves have many friends. Thus, there is a propensity for positive assortative matching—something we would like to explore further in future work.

Our analysis is related to, but differs from, the work by Jackson et al [5] in important ways. One particular source of difference in our conclusions is the nature of punishment following violation of a convention. In their paper, the punishment is carried out by a subset of neighbors that are common between the supposed-to-be giver and receiver of favors. This leads the stability condition giving rise to a high-support structure. In our model, on the other hand, the punishment comes from the neighbors of the violator of the convention. This assumption is more suitable for studying "global norms," i.e. norms pervading in the entire community, as opposed to the "local norms," which Jackson et al capture. Given that our punishment strategy is stronger than theirs, our results about the efficient network not being stable are also stronger, and will carry over to a set-up with local enforcement of norms.

Our paper also contributes to the relationship between pairwise stability and Nash equilibria studied by Calvó-Armengol and Ilkilic [2]. In our framework, with link deletion as the set of permissible strategies, there is an equivalence between the two concepts. However, it can be shown that this does not generalize to addition of links because of super-modularity—while it may benefit a player to add several links, each link may not, by itself, be worth adding.

Finally, our model provides a one-shot, simultaneous move characterization of the network formation game. In future work, we aim at studying the evolutionary stability of the network structure by starting with the initial network and presenting agents at random with link-deletion opportunities.

References

[1] Calvó-Armengol, A. Job contact networks. *Journal of Economic Theory 115* (2004), 191–206.

[2] Calvó-Armengol, A., and Ilkilic, R. Pairwise-stability and nash equilibria in network formation. *International Journal of Game Theory 38* (2009), 51–79.

[3] Goyal, S., and Joshi, S. Unequal connections. *International Journal of Game Theory 34* (2006), 319–349.

[4] Jackson, M., and Wolinsky, A. A strategic model of social and economic networks. *Journal of Economic Theory 71* (1996), 44–74.

[5] Jackson, M. O., Rodriguez-Barraquer, T., and Tan, X. Social capital and social quilts: Network patterns of favor exchange. *American Economic Review 102(5)* (2012), 1857–97.

[6] Jackson, M. O., and Rogers, B. The economics of small worlds. *Journal of the European Economic Association 3* (2005), 617–627.

[7] Johnson, C., and Giles, R. Spacial social networks. *Review of Economic Design, 5* (2000), 273–300.

[8] Neilson, W. S. The economics of favors. *Journal of Economic Behaviour and Orangization 39* (1999), 387–397.

6 Appendix

Proof Claim 1:

The expected gain of agent i from being in a network g is

$$\mathbf{P}\left(i \text{ needs a favor}\right) \cdot \mathbf{P}\left(\min_{k \in \mathcal{N}_i} c_k \leq c^*\right) \cdot v. \tag{6}$$

Firstly, recall that $\mathbf{P}\left(i \text{ needs a favor}\right) = 1/N$. Secondly, let $f\left(c_k\right)$ and $F\left(c_k\right)$ be the probability density function (pdf) and cumulative distribution function (cdf) of a random variable c_k, respectively. Then the cdf of the random variable

$$\min_{k \in \mathcal{N}_i} c_k,$$

is known to be

$$1 - \left[1 - F\left(s\right)\right]^{d_i}. \tag{7}$$

Hence, expected gain (6) can be written as

$$\frac{v}{N} \cdot \left(1 - \left[1 - F\left(s\right)\right]^{d_i}\right). \tag{8}$$

Because agent i performs the favor whenever she is the lowest cost person in j's neighbors and her cost is below c^*, the expected cost to agent i when $j \in \mathcal{N}_i$ needs a favor is

$$\mathbf{P}\left(j \text{ needs a favor}\right) \cdot \mathbf{P}\left(c_i \leq c^* \text{ and } c_i \leq \min_{k \in \mathcal{N}_i} c_k\right) \cdot E\left(c_i \middle| c_i \leq c^* \text{ and } c_i \leq \min_{k \in \mathcal{N}_i} c_k\right).$$

Again, $\mathbf{P}\left(j \text{ needs a favor}\right) = 1/N$. Next, notice that

$$\begin{aligned}
\mathbf{P}\left(c_i \leq c^* \text{ and } c_i \leq \min_{k \in \mathcal{N}_i} c_k\right) &= \mathbf{P}\left(c_i \leq \min_{k \in \mathcal{N}_i} c_k\right) \cdot \mathbf{P}\left(c_i \leq c^* \middle| c_i \leq \min_{k \in \mathcal{N}_i} c_k\right) \\
&= \frac{1}{d_j} \cdot \left(1 - \left[1 - F\left(c^*\right)\right]^{d_j}\right)
\end{aligned}$$

due to (7). And it can be shown that

$$E\left(c_i \middle| c_i \leq c^* \text{ and } c_i \leq \min_{k \in \mathcal{N}_i} c_k\right) = d_j \int_{-\infty}^{c^*} \frac{s \cdot f\left(s\right) \cdot \left[1 - F\left(s\right)\right]^{d_j - 1}}{1 - \left[1 - F\left(c^*\right)\right]^{d_j}} ds.$$

Therefore, the expected cost of agent i being in a network g with neighborhood $\mathcal{N}_i$ is

$$\frac{1}{N} \cdot \frac{1}{d_j} \cdot \left(1 - \left[1 - F\left(c^*\right)\right]^{d_j}\right) \sum_{j \in \mathcal{N}_i} \left\{ d_j \int_{-\infty}^{c^*} \frac{s \cdot f\left(s\right) \cdot \left[1 - F\left(s\right)\right]^{d_j - 1}}{1 - \left[1 - F\left(c^*\right)\right]^{d_j}} ds \right\}$$

or

$$\frac{1}{N} \sum_{j \in \mathcal{N}_i} \int_{-\infty}^{c^*} s \cdot f\left(s\right) \cdot \left[1 - F\left(s\right)\right]^{d_j - 1} ds. \tag{9}$$

Combining (8) and (9), the expected utility (benefits minus costs) of agent i being in a network g with neighborhood $\mathcal{N}_i$ is

$$EU^i\left(g, c^*, v\right) = \frac{v}{N} \cdot \left(1 - \left[1 - F\left(s\right)\right]^{d_i}\right) - \frac{1}{N} \sum_{j \in \mathcal{N}_i} \int_{-\infty}^{c^*} s \cdot f\left(s\right) \cdot \left[1 - F\left(s\right)\right]^{d_j - 1} ds,$$

which can be normalized to

$$EU^i\left(g, c^*, v\right) = v \cdot \left(1 - \left[1 - F\left(s\right)\right]^{d_i}\right) - \sum_{j \in \mathcal{N}_i} \int_{-\infty}^{c^*} s \cdot f\left(s\right) \cdot \left[1 - F\left(s\right)\right]^{d_j - 1} ds. \tag{10}$$

Recall that c_k is assumed to have a uniform distribution on the unit interval, $[0, 1]$. Hence, Equation 10 can be written as:

$$\begin{aligned} EU^i\left(g, c^*, v\right) =& v\left[1 - \left(1 - c^*\right)^{d_i}\right] \\ & - \sum_{j \in \mathcal{N}_i} \frac{1}{d_j} \left[\frac{1}{d_j + 1} - c^*\left(1 - c^*\right)^{d_j} - \frac{1}{d_j + 1}\left(1 - c^*\right)^{d_j + 1}\right]. \end{aligned}$$

Proof Lemma 1:

If agent $j \in \mathcal{N}_i(g)$ has d_j neighbors rather than $d_j - 1$, the benefit to agent i does not change, but the cost bear by agent i from having agent j as a neighbor does.
If agent i is better off when his neighbor j has d_j neighbors, it has to be that, from Equation 9:

$$\int_{-\infty}^{c^*} s \cdot f(s) \cdot [1 - F(s)]^{d_j - 1} ds \geq \int_{-\infty}^{c^*} s \cdot f(s) \cdot [1 - F(s)]^{d_j} ds$$

which is obviously the case. □

Proof Lemma 2:

Rearrange Equation 4 into:

$$2v\left[1-(1-c^*)^k\right] > k\,(c^*)^2 \tag{11}$$

and let $f(c^*) = 2v\left[1-(1-c^*)^k\right]$ and $g(c^*) = k\,(c^*)^2$. To show (4), it is enough to show that there exist such c^* satisfying equation (11). It is obvious that $f(0) = g(0) = 0$, $f(1) = 2v$, and $g(1) = k$. Since $f'(c^*) = 2vk\,(1-c^*)^{k-1}$ and $g'(c^*) = 2kc^*$, $f'(0) = 2vk > g'(0) = 0$ for any $v > 0$. Furthermore, $f''(c^*) = -2vk\,(k-1)\,(1-c^*)^{k-2} \leq 0$ and $g''(c^*) = 2k > 0$ for all $c^* \in (0,1]$. Therefore if $2v > k$, equation (11) holds for all $c^* \in (0,1]$. When $2v \leq k$, $f(c^*)$ and $g(c^*)$ intersect at a point $\tilde{c}$ such that $0 < \tilde{c} \leq 1$ and equation (11) is true for all $c^* \in (0,\tilde{c})$.□

Proof Theorem 1

We first decompose the network into N subnetworks that are star networks. Each star subnetwork admits one of the agents as the star, and the neighbors of this agent as the satellites of the star. From Lemma 2, we know that for each subnetwork $g_{S_{i,d_i}}$, there exists a convention, call it $c^*_{S_{i,d_i}}$, for which the star agent receives a strictly positive expected utility. This convention is not unique, and we denote $\overline{c}^*_{S_{i,d_i}} = \sup\left\{c^*_{S_{i,d_i}}\right\}$. Any convention $c^*_{S_{i,d_i}}$ in the interval $(0,\overline{c}^*_{S_{i,d_i}})$ supports $(g_{S_{i,d_i}}, c^*_{S_{i,d_i}}; v)$ as Participation Compatible (Lemma 2). Following Lemma 1, we know that, since $EU^i(g_{S_{i,d_i}}, c^*_{S_{i,d_i}}; v) > 0$, then $EU^i(g, c^*_{S_{i,d_i}}; v)$ is also strictly positive (where g is the original network with N agents) since $d_j(g) \geq d_j(S_{i,d_i})$ for all $j \in \mathcal{N}_i(g)$.

The intersection of the N intervals $(0,\overline{c}^*_{S_{i,d_i}})$ for all $i \in N$ is the set conventions that are participation compatible for all agents. Alternatively, by taking the minimum of all $\overline{c}^*_{S_{i,d_i}}$ that have been identified for all the star subnetworks, we can find this intersection. Formally, denote $c^*_P = \min_{i\in\mathcal{N}} \overline{c}^*_{S_{i,d_i}}$. Agents can always adopt a convention $c^*_p \in (0, c^*_P)$ which makes the system Participation Compatible. □

Proof Theorem 2:

Consider agent i in network g. If agent i has no incentives to delete a link, then it has to be that for any $j \in \mathcal{N}_i(g)$, the following inequality holds.

$$EU^i(g, c_i^*; v) \geq EU^i(g \setminus \{i,j\}, c_i^*; v)$$
$$= vc_i^* (1 - c_i^*)^{d_i - 1} - \frac{1}{d_j}\left[\frac{1}{d_j+1} - c^* (1 - c_i^*)^{d_j} - \frac{1}{d_j+1}(1 - c_i^*)^{d_j+1}\right] \geq 0.$$

Let

$$f(c_i^*) = vc_i^* (1 - c_i^*)^{d_i - 1}$$

and

$$g(c_i^*) = \frac{1}{d_j}\left[\frac{1}{d_j+1} - c_i^* (1 - c_i^*)^{d_j} - \frac{1}{d_j+1}(1 - c_i^*)^{d_j+1}\right].$$

Then $f(0) = g(0) = f(1) = 0$ and $g(1) = \frac{1}{d_j(d_j+1)}$.

Because

$$f'(c_i^*) = -d_i v (1 - c_i^*)^{d_i - 2}\left(c_i^* - \frac{1}{d_i}\right)$$

and

$$g'(c_i^*) = c_i^* (1 - c_i^*)^{d_j - 1},$$

$f'(0) = v > g'(0) = 0$ for all $v > 0$.

Moreover, $f(c_i^*)$ is maximized at $c_i^* = 1/d_i$ and $g'(c_i^*) > 0$ for all $c_i^* \in (0,1)$. Hence $f(c_i^*)$ and $g(c_i^*)$ intersect at $\tilde{c}_{ij}^*$ such that $0 < \tilde{c}_{ij}^* < 1$. Condition $EU^i(g, c_i^*; v) \geq EU^i(g \setminus \{i,j\}, c_i^*; v)$ is satisfied for all $c_i^* \in [0, \tilde{c}_{ij}^*]$.
We can find $\tilde{c}_{ij}^*$ for all $j \in \mathcal{N}_i$ in this way. If we define $\tilde{c}_i^* = \min_{j \in \mathcal{N}_i}\{\tilde{c}_{ij}^*\}$, the RPS condition is satisfied for all j when $c_i^* \in [0, \tilde{c}_i^*]$. Using the same procedure, we can find $\tilde{c}_i^*$ for all $i \in \mathcal{N}$. Let $c_{RPS}^* = \min_{i \in \mathcal{N}}\{\tilde{c}_i^*\}$ then a system $(g, c^*; v)$ is RPS for any $c^* \in [0, c_{RPS}^*]$.□

Proof Proposition 1:

If $(g, c^*; v)$ is Restricted Pairwise Stable, we have a convention $c^* \in [0, c^*_{RPS}]$ such that, for any agent $i \in \mathcal{N}$:

$$vc^*(1-c^*)^{d_i} - \frac{1}{d_j}\left[\frac{1}{d_j+1} - c^*(1-c^*)^{d_j} - \frac{1}{d_j+1}(1-c^*)^{d_j+1}\right] \geq 0$$

Since this condition must be true for all $j \in \mathcal{N}_i$ we can simply add up all d_i of the conditions such that

$$\sum_{j\in\mathcal{N}_i}\left\{vc^*(1-c^*)^{d_i} - \frac{1}{d_j}\left[\frac{1}{d_j+1} - c^*(1-c^*)^{d_j} - \frac{1}{d_j+1}(1-c^*)^{d_j+1}\right]\right\} \geq 0$$

which then becomes

$$d_i vc^*(1-c^*)^{d_i} \geq \sum_{j\in\mathcal{N}_i}\frac{1}{d_j}\left[\frac{1}{d_j+1} - c^*(1-c^*)^{d_j} - \frac{1}{d_j+1}(1-c^*)^{d_j+1}\right]. \quad (12)$$

To prove that the system is also Participation Compatible, it suffices to show that for all d_i and $c^* \in [0, c^*_{RPS}]$, the following inequality holds:

$$v[1-(1-c^*)^{d_i}] > d_i vc^*(1-c^*)^{d_i}. \quad (13)$$

because of equation 3. Equation 13 can be rewritten as:

$$1 > (d_i c^* + 1)(1-c^*)^{d_i}.$$

Note that we only need to consider $c^* \in (0, c^*_{RPS}]$ since the system is Restricted Pairwise Stable and Participation Compatibility only requires c^* to be in the interval $(0, 1]$.
To verify this, let

$$f(c^*) = \frac{1}{(1-c^*)^{d_i}} \text{ and } g(c^*) = d_i c^* + 1.$$

Note that $f(0) = g(0) = 1,\ f(1) = \infty$, and $g(1) = 1 + d_i$. Since

$$f'(c^*) = \frac{d_i}{(1-c^*)^{d_i+1}} \text{ and } g'(c^*) = d_i$$

$f'(c^*) > g'(c^*)$ for all $c^* \in (0, c^*_{RPS}]$. Getting second derivatives of f and g

$$f''(c^*) = \frac{d_i(d_i+1)}{(1-c^*)^{d_i+2}} \text{ and } g''(c^*) = 0.$$

Therefore $f''(c^*_{RPS}) > g''(c^*)$ for all $c^* \in (0, c^*_{RPS}]$, equation 13 is true.
□

Proof Theorem 3:

Consider a given set of values $\{g, c^*, v\}$. Suppose that under network g, agent i has neighbors $\{1, 2, ..., d_i\}$. Take an arbitrary subset $\mathbb{k}$ of size $K(\leq d_i)$ of these neighbors. Denote $EU^i(g)$ agent i's expected utility under network g (at the fixed c^* and v) and let $EU^i(g \setminus \mathbb{k})$ denote the expected utility from the subnetwork in which agent i has dropped $\mathbb{k}$ from her set of neighbors.
We claim that if $EU^i(g \setminus \mathbb{k}) \geq EU^i(g)$ then there exists $j \in \mathbb{k}$ such that $EU^i(g \setminus \{i, j\}) \geq EU^i(g)$.
This claim asserts that if agent i is better off by dropping a subset $\mathbb{k}$ of her neighbors, then she is also better of dropping some neighbor $j \in \mathbb{k}$.
Let d_j denote the number of j's neighbor for $j \in \mathbb{k}$. Without any loss of generality we can relabel i's neighbors such that the agents in $\mathbb{k}$ are $\{1, 2, ..., K\}$ where $d_1 \leq d_2 \leq ... \leq d_K$. The remaining $d_i - K$ neighbors can take names $K+1, ..., d_i$ in any arbitrary order. Note that $EU^i(g)$ can be written as $B(g) - \sum_{j=1}^{d_i} C_j$ where $B(g)$ denotes the expected benefit to i from having d_i friends and C_j denotes the expected cost to i of having j as a friend. Note that the term $B(g)$ depends only on the number of i's friends (i.e. on d_i only) while each C_j depends on the number of friends of each friend, i.e. on the d_js. Under our convention, we have $C_1 \geq C_2 \geq ... \geq C_K$ since the expected cost of having someone as friend is inversely related to the number of *their* friends. Looking at the benefits term, observe that

$$B(g) = v[1 - (1 - c^*)^{d_i}]$$

is an increasing and strictly concave function in d_i which means that if we start reducing the number of agent i's friends from d_i to $d_i - K$, $B(.)$ will decrease faster and faster with each deletion of a friend. In particular we have

$$B(g) - B(g \setminus \mathbb{k}) > K \cdot [B(g) - B(g \setminus \{i, 1\})]. \tag{14}$$

Suppose that

$$EU^i(g \setminus \mathbb{k}) \geq EU^i(g)$$

i.e.

$$B(g \setminus \mathbb{k}) - \sum_{j=K+1}^{d_i} C_j \geq B(g) - \sum_{j=1}^{d_i} C_j$$

i.e.

$$\sum_{j=1}^{K} C_j \geq B(g) - B(g \setminus \Bbbk).$$

This implies, given (14), that

$$\sum_{j=1}^{K} C_j > K \cdot [B(g) \setminus B(g \setminus \{i, 1\})]$$

or

$$\frac{1}{K}\sum_{j=1}^{K} C_j > B(g) - B(g \setminus \{i, 1\}).$$

Since $C_1 = \max\{C_1, ..., C_K\}$ it follows that

$$C_1 > B(g) - B(g \setminus \{i, 1\}).$$

That is, if starting from network g, dropping some subset of $\Bbbk$ neighbors is a profitable deviation for i, then dropping the least desirable of these $\Bbbk$ neighbors is also a profitable deviation.□

Proof Proposition 2:

We need to find c^* that maximize the sum of expected utilities

$$f(c^*) = \sum_{i=1}^{N} EU^i (g, c^*; v). \tag{15}$$

The first order condition is

$$f'(c^*) = \sum_{i=1}^{n} \left[d_i v (1 - c^*)^{d_i - 1} - \sum_{j \in \mathcal{N}_i} c^* (1 - c^*)^{d_j - 1} \right] = 0. \tag{16}$$

The term $c^* (1 - c^*)^{d_k - 1}$, $k = 1, \cdots, N$ appears d_k times in equation (16). Grouping those terms together for every agent yields

$$f'(c^*) = \sum_{i=1}^{n} \left[d_i (v - c^*) (1 - c^*)^{d_i - 1} \right] = 0. \tag{17}$$

From equation 17, it is clear that $c^* = 1$ and v are the only roots in $[0, 1]$ when $v \in [0, 1]$.
The second order condition is

$$f''(c^*) = d_i \sum_{i=1}^{N} \left[(d_i c^* + v - d_i v - 1)(1 - c^*)^{d_i - 2} \right],$$

which is at $c^* = v$,

$$-d_i \sum_{i=1}^{N} (1 - v)^{d_i - 1} < 0.$$

Thus equation (15) is maximized at $c^* = v$ when $v \in (0, 1)$. However, if $v \geq 1$, the sum of expected utilities is maximized at $c^* = 1$ since $f'(c^*) \geq 0$ on $[0, 1]$. □

Proof Proposition 3:

Let $g_{S_{i,N-1}}$ be the star network where i is the agent at the centre of the star and $N - 1$ is the number of satellite agents. For the agent at the centre, the Restricted Pairwise Stability condition, derived from equation 4, reduces to

$$2v^2(1 - v)^{N-2} + 2v(1 - v) + (1 - v)^2 \geq 1$$

$$(1 - v)^{N-2} \geq \frac{1}{2}$$

$$v \leq 1 - (\frac{1}{2})^{\frac{1}{N-2}}.$$

Since $\lim_{N \to \infty} 1 - (\frac{1}{2})^{\frac{1}{N-2}} = 0$ and $\lim_{N=2} 1 - (\frac{1}{2})^{\frac{1}{N-2}} = 1$, the next step is to show that no satellite agent has any incentives to delete links for any value of $v \in (0, 1)$.
The Restricted Pairwise Stability condition for satellites, when $c^* = v$ is:

$$(N - 1)v^2 \geq \frac{1}{N} - v(1 - v)^{N-1} - \frac{1}{N}(1 - v)^N$$

which can be rewritten as

$$N(N-1)v^2 + Nv(1-v)^{N-1} + (1-v)^N \geq 1.$$

Let $f(v) = N(N-1)v^2 + Nv(1-v)^{N-1} + (1-v)^N$. For any $N \geq 2$, we have $\lim_{v\to 0} f(v) \to= 1$ and $\lim_{v\to 1} f(v) \to= N(N-1)$. The next step is then to show that $f(v)$ is increasing on $(0,1)$. This amounts to show that:

$$2vN(N-1) + (-1)N(1-v)^{N-1} + N(1-v)^{N-1} - Nv(1-v)^{N-2} > 0.$$

Rearranging, the condition becomes

$$2(N-1) > (1-v)^{N-2}$$

which is always satisfied for $N \geq 2$. Hence, the Restricted Pairwise Stability condition for the centre of the star is both, necessary and sufficient, to insure that the system is Restricted Pairwise Stable.$\square$

Proof Proposition 4:

For regular networks, the Restricted Pairwise Stability condition for any agent at $c^* = v$ can be rewritten as:

$$n^2v^2 + (n-1)v + 1 \geq (1-v)^{1-n}.$$

Define

$$f(v) = n^2v^2 + (n-1)v + 1 \text{ and } g(v) = \frac{1}{(1-v)^{n-1}}.$$

Note that $f(0) = g(0) = 1$, $f(1) = n(n+1)$, and $g(1) = \infty$. Furthermore, $f'(v) = 2n^2v + n - 1 > 0$ and $g'(v) = (n-1)(1-v)^{-n} > 0$ for all $n \geq 2$ and $v \in (0,1)$. Because $f''(v) = 2n^2$ and $g''(v) = n(n-1)(1-v)^{-n-1}$, $f''(v) \geq g''(v)$ for all $v \in (0, 1 - [(n-1)/2n]^{1/n+1}]$.$\square$

Proof Lemma 4:

Traditional Social Welfare, $SW(g, c^*; v)$ can be defined as:

$$SW(g, c^*; v) = \sum_{i\in\mathcal{N}} \{Pr(i \text{ needs a favor}) \cdot Pr(i \text{ receives a favor}) \cdot (v - \text{ Exp. Social Cost of doing a favor for } i)\}$$

Given $\mathcal{N}_i$, the set of i's neighbors, let $c_{\mathcal{N}_i}$, denote the random variable $\min_{j\in\mathcal{N}_i}\{c_j\}$. The cdf and pdf of $c_{\mathcal{N}_i}$ are denoted by F_i and f_i, respectively. We can then write:

$$\begin{aligned} SW(g, c^*; v) &= \sum_{i=1}^{N} \frac{1}{N} \cdot F_i(c^*) \cdot [v - E(c_{\mathcal{N}_i} | c_{\mathcal{N}_i} \leq c^*)] \\ &= \frac{1}{N} \sum_{i=1}^{N} F_i(c^*) \cdot \left[v - \int_0^{c^*} c f_i(c) dc\right] \end{aligned}$$

Note that when $c^* \leq v$, we have $[v - E(c_{\mathcal{N}_i} | c_{\mathcal{N}_i} \leq c^*)] \geq 0$.
Now consider a move from g to g' by increasing the size of agent i's neighborhood, i.e., $\mathcal{N}_i' \supset \mathcal{N}_i$. Then, we have:

$$F_k'(c^*) \geq F(c^*) \text{ and } E(c_{\mathcal{N}_k'} | c_{\mathcal{N}_k'}) \leq c^*)] \leq E(c_{\mathcal{N}_k} | c_{\mathcal{N}_k} \leq c^*)],$$

for all agents $k \in \left\{\left\{\mathcal{N}_i' \setminus \mathcal{N}_i\right\} \cup \{i\}\right\}$.

For any other agent $j \notin \left\{\left\{\mathcal{N}_i' \setminus \mathcal{N}_i\right\} \cup \{i\}\right\}$, $d_j(g) = d_j(g')$. It follows that $SW(g', c^*; v) \geq SW(g, c^*; v)$, i.e., increasing an agent's neighborhood size increases social welfare. Since this is true for any arbitrary agent and any arbitrary starting point, the complete network maximizes social welfare so long as $c^* \leq v$.$\square$

Proof Proposition 7:

It suffices to show that when $c^* \leq v$, adding (deleting) a link improves (reduces) favor efficiency. Also, since $c^* \leq v$ implies $\varepsilon_i = 0$ for all i, maximizing W is the same as maximizing δ. Let $g' = g \cup \{i, j\}$. Note that for $k \neq i, j$ we have $d_k(g') = d_k(g)$ and therefore $Pr(c_{\mathcal{N}_{k(g')}} \leq c^*) = Pr(c_{\mathcal{N}_{k(g)}} \leq c^*)$, i.e., $\delta_k' = \delta_k$.
At the same time, for i (and similarly for j) we have $Pr(c_{\mathcal{N}_{i(g')}} \leq c^*) > Pr(c_{\mathcal{N}_{i(g)}} \leq c^*)$, i.e., $\delta_i' > \delta_i$ since the minimum of i.i.d. draws over a greater number of draws have a greater probability of being smaller than any given threshold.
Hence, we have:

$$\delta = \frac{1}{N}\{\sum_{k\neq i,j} \delta_k + \delta_i + \delta_j\}$$

and

$$\delta' = \frac{1}{N}\{\sum_{k\neq i,j} \delta'_k + \delta'_i + \delta'_j\}.$$

And as argued above, $\delta'_k = \delta_k$ for $k \neq i, j$, $\delta'_i > \delta_i$ and $\delta'_j > \delta_j$. □