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Abstract: We derive necessary and sufficient conditions for data sets composed of state-contingent prices and consumption to be consistent with two prominent models of decision making under uncertainty: variational preferences and smooth ambiguity. The revealed preference conditions for subjective expected utility, maxmin expected utility, and multiplier preferences are characterized as special cases. We implement our tests on data from a portfolio choice experiment.

JEL Classifications: D1, D8.

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1. Introduction

Recent developments in behavioral economics and decision theory have led to a series of increasingly general models of decision making under risk and uncertainty, e.g., Choquet expected utility (Schmeidler, 1989), maxmin expected utility (Gilboa and Schmeidler, 1989), cumulative prospect theory (Tversky and Kahneman, 1992), biseparable preferences (Ghirardato, Maccheroni, and Marinacci, 2004), smooth ambiguity (Klibanoff, Marinacci, and Mukerji, 2005), variational preferences (Maccheroni, Marinacci, and Rustichini, 2006), subjective expected uncertain utility (Gul and Pesendorfer, 2008), vector expected utility (Siniscalchi, 2009), and uncertainty averse preferences (Cerreia-Vioglio *et al.*, 2011), to name a few.[1] This paper addresses the issue of testability in two of the most prominent of these models: variational preferences and smooth ambiguity.

More precisely, we adopt the revealed preference approach pioneered by Samuelson (1938, 1948) and Afriat (1967). We derive necessary and sufficient conditions for data sets composed of state-contingent prices and consumption to be consistent with variational preferences (Maccheroni, Marinacci, and Rustichini, 2006) and smooth ambiguity (Klibanoff, Marinacci, and Mukerji, 2005). We then characterize multiplier preferences (Hansen and Sargent, 2001), maxmin expected utility (Gilboa and Schmeidler, 1989), and subjective expected utility (Savage, 1954) as special cases.[2] These models of decision making under uncertainty are central in a wide range of economic applications.

Let us contrast our approach with more traditional methods in applied econometrics and experimental economics, e.g., in Ahn *et al.* (2011), and Hey and Pace (2012). To check whether the observed choices are consistent with a particular model of decision making, the more conventional approach consists of postulating specific functional forms, deriving optimal choices, and estimating an econometric specification assuming that observed choices are optimal subject to errors. For instance, when considering the multiple prior model, Hey and Pace (2012) assume that the Bernoulli utility function is either CRRA or CARA, that the set of priors is a truncation from below of the probability simplex, and that errors are normally distributed with zero mean and constant variance. These specifications may fail to rationalize the

[1] See Gilboa and Marinacci (2011) and Wakker (2010) for recent surveys.

[2] Note that mean-variance preferences (Markowitz, 1952, 1959; Tobin, 1958) are also nested.

observed choices. This does not imply, however, that other specifications of multiple prior preferences cannot rationalize the data. In sharp contrast, our tests are fully *nonparametric* and *exact*, i.e., if the observed choices fail to satisfy our tests, then they *cannot* be consistent with the maximization of multiple prior preferences.

We illustrate the problem further with the help of a simple example, inspired by the experiment in Ahn *et al.* (2011). There are three states of the world, s_1, s_2, and s_3, and a single consumption good (say, money) in each state. The probability of state s_1 is known to be 1/3, while the probabilities of the other two states are unknown. An individual must choose a contingent consumption plan, (x_1, x_2, x_3), where x_i refers to consumption of the good in state s_i, from two different budget sets. The first budget set is $B := \{(x_1, x_2, x_3) : x_1 + (4/5)x_2 + 2x_3 \leq 1\}$, while the second is $B' := \{(x_1, x_2, x_3) : 2x_1 + 2x_2 + x_3 \leq 3\}$. Suppose that we observe the individual choosing $(1, 0, 0)$ from budget set B and $(0, 1, 1)$ from budget set B'. Notice that $(0, 1, 1)$ is chosen when $(1, 0, 0)$ is affordable, while $(0, 1, 1)$ is not affordable when $(1, 0, 0)$ is chosen. The choices therefore obey the Generalized Axiom of Revealed Preference (GARP), and by Afriat's Theorem there exists a monotonic preference ordering that rationalizes the data. The question we address in this paper is whether these observations are consistent with a particular form of preference maximization. For example, are they consistent with subjective expected utility? Or with an ambiguity averse preference like maxmin expected utility? In other words, can we find particular preferences that exactly generate the observed choices? Conversely, can we rule out particular preferences?

Let us return to the example. We first argue that the data set is *inconsistent* with the individual maximizing some subjective expected utility. Notice that $(0, 1, 0)$ is in B and that $(1, 0, 1)$ is in B'. So it must be that $(1, 0, 0)$ is preferred to $(0, 1, 0)$ and that $(0, 1, 1)$ is preferred to $(1, 0, 1)$. Moreover, assuming that preferences are monotonic, $(1, 0, 0)$ is *strictly* preferred to $(0, 1, 0)$.[3] Therefore, we have that $(1, 0, 0)$ is strictly preferred to $(0, 1, 0)$, while $(0, 1, 1)$ is preferred to $(1, 0, 1)$. This is a violation of the *sure-thing principle*,[4] and therefore the data set cannot be consistent with subjective

[3] Notice that $(1/19, 1 + 1/19, 1/19) \gg (0, 1, 0)$ is in B. Consequently, if the decision maker is indifferent between $(1, 0, 0)$ and $(0, 1, 0)$, we would have that $(1/19, 1 + 1/19, 1/19)$ is strictly preferred to $(1, 0, 0)$ by monotonicity, a contradiction.

[4] When choosing between $(1, 0, 0)$ and $(0, 1, 0)$ in B, consuming 0 in state s_3 is a sure thing; similarly, consuming 1 in state s_3 is a sure thing when choosing between $(0, 1, 1)$ and $(1, 0, 1)$ in B'. If the agent strictly prefers $((1, 0), 0)$ to $((0, 1), 0)$, then $((0, 1), 1)$ cannot be preferred to $((1, 0), 1)$.

expected utility. There is no utility function and probability distribution over states that rationalizes the observed choices.

We next argue that the data set is *consistent* with the individual maximizing some maxmin expected utility. To see this, consider the utility function $u(x) = x$ and the set of priors $\Pi = [0, 2/3]$, where $\pi \in \Pi$ is the probability of state s_3. It is straightforward to verify, using linear programming techniques, that $(1, 0, 0)$ is a solution to the optimization problem $\max_{x \in B} \min_{\pi \in \Pi}(1/3)x_1 + (2/3 - \pi)x_2 + \pi x_3$, while $(0, 1, 1)$ is a solution to the optimization problem $\max_{x \in B'} \min_{\pi \in \Pi}(1/3)x_1 + (2/3 - \pi)x_2 + \pi x_3$.

In this simple example, we are able to explicitly reject the hypothesis that the data set is consistent with subjective expected utility.[5] At the same time, we cannot rule out the hypothesis that the data set is consistent with maxmin expected utility. In general, however, we would need to consider *all* utility functions and *all* sets of priors, to derive *all* sets of optimal choices, and then to determine whether or not the observed data are contained in these sets. Clearly, this is not feasible. Instead, we follow the Afriat approach and derive fully nonparametric tests for consistency in the presence of finite data.

2. Ambiguity Revealed

2.1 Preliminaries

Consider an economy with commodity space $X = \mathbb{R}^\ell_+$ and finite state space S, with generic elements x and s, respectively. Trading takes place before the realization of a state in a complete market. Let $x(s)$ denote consumption of the ℓ goods in state s, with corresponding state-contingent price vector $p(s) \in \mathbb{R}^\ell_{++}$. Let $\Delta(S)$ denote the set of probability distributions over S, with generic element π. In such an economy, the choice of a consumer corresponds to the choice of an act $x : S \to X$. Let $\mathbb{X}$ denote the set of all acts, and let $\succcurlyeq \subseteq \mathbb{X} \times \mathbb{X}$ be the consumer's preference ordering over acts.

[5] Notice that our example makes use of consumption plans which correspond to the acts constituting the decision maker's choice set in the classic three-color Ellsberg (1961) experiment. In our case, however, the agent chooses from a standard (convex) budget set and therefore can hedge away ambiguity *completely* by choosing plans where $x_2 = x_3$. This setting makes it possible to draw richer conclusions than in the classical setting where the decision maker is not free to choose convex combinations.

Suppose that we observe the consumer's choices over a finite number of periods, that is, we have the data set $(x_t, p_t)_{t \in T}$. The main question we address is whether the data set is consistent with some form of preference maximization. In other words, does there exist a preference relation $\succcurlyeq$ such that $x_t \succcurlyeq x$ for all $x \in B(x_t, p_t) := \{x : p_t \cdot x \leq p_t \cdot x_t\}$, the budget set at (x_t, p_t), for all $t \in T$?

Naturally, without further assumptions, *any* data set is consistent with preference maximization. To see this, simply assume that the consumer is indifferent between all acts. Throughout the paper, we make a series of assumptions on the consumer's preferences: completeness, transitivity, continuity, monotonicity, and non-degeneracy (i.e., axioms A.1, A.3, A.4, and A.6 in Gilboa and Schmeidler, 1989).[6] These assumptions are standard and imply that the data set $(x_t, p_t)_{t \in T}$ is consistent with preference maximization if and only if it satisfies the Afriat inequalities, namely, if and only if there exist $(U_t, \lambda_t)_{t \in T}$, with $(U_t, \lambda_t) \in \mathbb{R} \times \mathbb{R}_{++}$, such that $U_{t'} \leq U_t + \lambda_t p_t \cdot (x_{t'} - x_t)$ for all $(t, t') \in T \times T$.[7] Consequently, if the data set satisfies the Afriat inequalities, then there exists a utility function $U : \mathbb{X} \to \mathbb{R}$ that rationalizes the observed choices. The purpose of this paper is to establish revealed preference characterizations for specific functional forms of U, e.g., when there exists a Bernoulli utility function $u : X \to \mathbb{R}$ and a prior $\pi \in \Delta(S)$ such that $U(x) = \sum_s \pi(s) u(x(s))$ for all $x \in \mathbb{X}$.

2.2 Variational Preferences

The class of variational preferences (Maccheroni, Marinacci, and Rustichini, 2006) is a broad class that incorporates ambiguity and generalizes multiple prior preferences (Gilboa and Schmeidler, 1989).[8] It also includes, as special cases, multiplier preferences (Hansen and Sargent, 2001) and mean-variance preferences (Markowitz, 1952, 1959; Tobin, 1958), two prominent models in macroeconomics and finance.

[6] More precisely, we assume two axioms of monotonicity. The first axiom is classical in decision theory and states that if $x(s) \succcurlyeq y(s)$ for all $s \in S$, then $x \succcurlyeq y$; the act $x(s)$ (resp., $y(s)$) is the constant act that gives $x(s)$ (resp., $y(s)$) in all states. The second axiom is specific to consumer theory and states that if $x(s) \gg y(s)$, then $x(s) \succ y(s)$.

[7] See Afriat (1967), Diewert (1973, 2012), Varian (1982), and Fostel, Scarf, and Todd (2004) for proofs of Afriat's Theorem. The revealed preference approach has been adopted in a variety of settings, for example, firm production (Hanoch and Rothschild, 1972; Varian, 1984), consumer demand (Varian, 1983a), investor behavior (Varian, 1983b), risk (Green and Srivastava, 1986; Varian, 1988), intertemporal choice (Browning, 1989; Crawford, 2010), collective decision making (Cherchye, De Rock, and Vermeulen, 2007), and the demand for characteristics (Blow, Browning, and Crawford, 2008), to name a few.

[8] Multiple prior preferences are also known as maxmin expected utility.

A decision maker with variational preferences evaluates the act $x \in \mathbb{X}$ as

$$\min_{\pi \in \Delta(S)} \sum_{s \in S} \pi(s) u(x(s)) + c(\pi),$$

where $u : X \to \mathbb{R}$ is a utility function and $c : \Delta(S) \to \mathbb{R}_+ \cup \{+\infty\}$ is a grounded, lower semi-continuous, and convex function. The function c can be viewed as an index of ambiguity aversion: the lower is $c(\pi)$, the higher is the aversion towards ambiguity. Multiple prior preferences and multiplier preferences then correspond to specific choices of c, namely, the indicator function of a closed convex set and a relative entropy measure, respectively.

The axioms of weak certainty independence and ambiguity aversion are central for the representation of variational preferences. The weak certainty independence axiom is a weakening of the classical independence axiom and essentially requires independence only when acts are mixed with *constant* acts for fixed mixing coefficients. Ambiguity aversion states that if a decision maker is indifferent between two acts, then he prefers mixtures of these two acts over either of them. This reflects the desire of the decision maker to hedge against ambiguity.

Accordingly, we say that the data set $(x_t, p_t)_{t \in T}$ is consistent with the maximization of variational preferences if there exist an increasing and concave utility function $u : X \to \mathbb{R}$ and a grounded, lower semi-continuous, and convex function $c : \Delta(S) \to \mathbb{R}_+ \cup \{+\infty\}$ such that $\min_{\pi \in \Delta(S)} \left(\sum_{s \in S} \pi(s) u(x_t(s)) + c(\pi) \right) \geq \min_{\pi \in \Delta(S)} \left(\sum_{s \in S} \pi(s) u(x(s)) + c(\pi) \right)$ for all $x \in B(x_t, p_t)$, for all $t \in T$. It is worth stressing that we assume a concave utility function to reflect risk aversion. While this assumption is standard, it is not without loss of generality, as we show in the conclusion.

Theorem 1 *Let $(x_t, p_t)_{t \in T}$ be the data set. The following statements are equivalent:*

1. *The data set $(x_t, p_t)_{t \in T}$ is consistent with the maximization of variational preferences.*

2. *There exist $(u_t, \pi_t, c_t, \lambda_t)_{t \in T}$, with $(u_t, \pi_t, c_t, \lambda_t) \in \mathbb{R}^{|S|} \times \Delta(S) \times \mathbb{R}_+ \times \mathbb{R}_{++}$ for each $t \in T$, such that*

$$u_{t'}(s') \leq u_t(s) + \lambda_t \frac{p_t(s)}{\pi_t(s)} (x_{t'}(s') - x_t(s)), \tag{V.1}$$

for all $(s, s', t, t') \in S \times S \times T \times T$, *and*

$$c_{t'} \geq c_t - \sum_{s \in S} u_t(s)(\pi_{t'}(s) - \pi_t(s)), \tag{V.2}$$

for all $(t, t') \in T \times T$.

The intuition behind Theorem 1 is simple. If the data set $(x_t, p_t)_{t \in T}$ is consistent with the maximization of variational preferences, then it must be that x_t is a solution to the problem $\max_{x \in B(p_t, x_t)} \min_{\pi \in \Delta(S)} \sum_s \pi(s)u(x(s)) + c(\pi)$. From Fan's minimax theorem, there exists $\pi_t \in \Delta(S)$ such that (x_t, π_t) is a saddle point, i.e.,

$$\sum_{s \in S} \pi(s)u(x_t(s)) + c(\pi) \geq \sum_{s \in S} \pi_t(s)u(x_t(s)) + c(\pi_t) \geq \sum_{s \in S} \pi_t(s)u(x(s)) + c(\pi_t),$$

for all $x \in B(p_t, x_t)$, for all $\pi \in \Delta(S)$. The first inequality directly implies condition (V.2). As for the second, it implies that x_t maximizes the consumer's expected utility given the belief π_t, and therefore condition (V.1) follows from classical optimality conditions for convex problems. Conversely, if the conditions (V.1) and (V.2) are satisfied, then we can construct an increasing, continuous, and concave utility function u and a grounded, continuous, and convex function c such that (u, c) rationalize the data, as in Afriat (1967).

The conditions (V.1) and (V.2) exhaust the observable implications of maximizing variational preferences. To test whether a data set satisfies (V.1) and (V.2) requires that we make use of *nonlinear* programming techniques, which are computationally difficult. To circumvent this difficulty, we can fix a set $\{\pi_t : t \in T\}$, solve for $(u_t, \lambda_t, c_t)_{t \in T}$ given the set $\{\pi_t : t \in T\}$, and then perform a grid or random search over the space $\Delta(S)^{|T|}$. This simplifies the computational problem to a linear program, the techniques for which are well established. However, we hasten to stress that the computational difficulties remain formidable, as we will see in the implementation section.

We next consider the special cases of multiple prior preferences and multiplier preferences. To obtain multiple prior preferences, we specify the ambiguity index c to be the indicator function of a non-empty, closed, and convex set $\Pi \subseteq \Delta(S)$ of priors, i.e., $c(\pi) = 0$ if $\pi \in \Pi$ and $c(\pi) = +\infty$ if $\pi \notin \Pi$. It immediately follows that a data set $(x_t, p_t)_{t \in T}$ is consistent with the maximization of multiple prior preferences if and only if there exist $(u_t, \pi_t, c_t, \lambda_t)_{t \in T}$, with $(u_t, \pi_t, c_t, \lambda_t) \in \mathbb{R}^{|S|} \times \Delta(S) \times \mathbb{R}_+ \times \mathbb{R}_{++}$

for each $t \in T$, such that (V.1) and (V.2) are satisfied, with the additional requirement that $c_t = 0$ for all $t \in T$.

Notice that when (V.1) and (V.2) are satisfied with $c_t = 0$ for all $t \in T$, the set of priors Π that we construct to rationalize the data is the convex hull of $\{\pi_t : t \in T\}$. This suggests that if there is a unique prior π that satisfies (V.1), then the data set is consistent with the maximization of Savage preferences, i.e., subjective expected utility. And indeed it is the case: the data set $(x_t, p_t)_{t\in T}$ is consistent with the maximization of Savage preferences if and only if there exist $(u_t, \pi_t, \lambda_t)_{t\in T}$, with $(u_t, \pi_t, \lambda_t) \in \mathbb{R}^{|S|} \times \Delta(S) \times \mathbb{R}_{++}$ for each $t \in T$, such that (V.1) holds, with the additional requirement that $\pi_t = \pi_{t'}$ for all $(t, t') \in T \times T$.[9]

To obtain multiplier preferences, we specify the ambiguity index c to be a positive multiple of the relative entropy of π with respect to some prior π^*, i.e., $c(\pi) = \theta R(\pi||\pi^*)$ with $\theta > 0$, $\pi^* \in \Delta(S)$, and $R(\pi||\pi^*) = \sum_{s\in S} \pi(s)[\ln \pi(s) - \ln \pi^*(s)]$. We have that a data set $(x_t, p_t)_{t\in T}$ is consistent with the maximization of multiplier preferences only if there exist $(u_t, \pi_t, \lambda_t)_{t\in T}$, with $(u_t, \pi_t, \lambda_t) \in \mathbb{R}^{|S|} \times \Delta(S) \times \mathbb{R}_{++}$ for each $t \in T$, and $(\theta, \pi^*) \in \mathbb{R}_{++} \times \Delta(S)$, such that (V.1) holds and

$$\pi_t(s) = \frac{\pi^*(s)e^{-\frac{u_t(s)}{\theta}}}{\sum_{\tilde{s}\in S} \pi^*(\tilde{s})e^{-\frac{u_t(\tilde{s})}{\theta}}},$$

for all $(s, t) \in S \times T$.[10] We next show that this implies that the data set is consistent with the maximization of subjective expected utility. To see this, define λ_t^* as $(1/\theta)\lambda_t \sum_{\tilde{s}} \pi^*(\tilde{s})e^{-\frac{u_t(\tilde{s})}{\theta}}$ so that condition (V.1) reads

$$u_{t'}(s') \leq u_t(s) + \frac{\lambda_t^*}{\pi^*(s)} \frac{\theta}{e^{-\frac{u_t(s)}{\theta}}} p_t(s)(x_{t'}(s') - x_t(s)).$$

Multiplying by $-1/\theta$ and taking the exponential of both sides, we obtain

$$\begin{aligned} \hat{u}_{t'}(s') &\geq \hat{u}_t(s) \exp\left(-\frac{\lambda_t^*}{\pi^*(s)} \frac{1}{\hat{u}_t(s)} p_t(s)(x_{t'}(s') - x_t(s))\right) \\ &\geq \hat{u}_t(s) \left(1 - \frac{\lambda_t^*}{\pi^*(s)} \frac{1}{\hat{u}_t(s)} p_t(s)(x_{t'}(s') - x_t(s))\right), \end{aligned}$$

with $\hat{u}_{\tilde{t}}(\tilde{s}) = e^{-u_{\tilde{t}}(\tilde{s})/\theta}$, where the second inequality follows from convexity of the exponential function. Finally, defining $u_t^*(s)$ as $-e^{-u_t(s)/\theta}$, we obtain

$$u_{t'}^*(s') \leq u_t^*(s) + \frac{\lambda_t^*}{\pi^*(s)} p_t(s)(x_{t'}(s') - x_t(s)).$$

[9] Green and Srivastava (1986) derive a similar condition for von Neumann-Morgenstern expected utility.

[10] This follows directly from the minimization with respect to π.

The data set is therefore consistent with the maximization of subjective expected utility (the prior is π^*). It is easy to see that the converse is also true, so that a data set is consistent with the maximization of multiplier preferences if and only if it is consistent with the maximization of Savage preferences. This result is not entirely new and has already been observed by Maccheroni, Marinacci and Rustichini (2006) and Strzalecki (2011). We have chosen this more indirect proof to illustrate how seemingly different Afriat inequalities may actually be equivalent.

2.3 Smooth Ambiguity

Smooth ambiguity (Klibanoff, Marinacci, and Mukerji, 2005) is another important class of preferences for decision making under uncertainty. This model assumes that a decision maker may form a range of predictions about future events and is uncertain about these predictions. As a concrete example, monetary authorities frequently use an array of models to predict future inflation and form assessments about the likelihood of each being the true model. A critical feature of smooth ambiguity is that it relaxes the reduction axiom of first and second order probabilities. According to this model, a decision maker evaluates an act $x \in \mathbb{X}$ as

$$\sum_{\pi} \phi \left(\sum_{s} \pi(s) u(x(s)) \right) \mu(\pi),$$

where $u : X \to \mathbb{R}$ is a utility function, π is a probability measure over S, μ is a probability measure over $\Delta(S)$, and $\phi : \mathbb{R} \to \mathbb{R}$ is a real-valued function. We may interpret $\mu(\pi)$ as the decision maker's subjective belief that the true model is π. Much in the same way that the function u captures the attitude of the decision maker towards risk, the function ϕ captures the attitude towards ambiguity. In particular, a concave ϕ characterizes ambiguity aversion. Smooth ambiguity includes, as a limiting case under infinite ambiguity aversion, the maxmin expected utility model of Gilboa and Schmeidler (1989).

We say that the data set $(x_t, p_t)_{t \in T}$ is consistent with the maximization of smooth preferences if there exist an increasing and concave function $u : X \to \mathbb{R}$, an increasing and concave function $\phi : \mathbb{R} \to \mathbb{R}$, a finite set of probability distributions $\Pi \subseteq \Delta(S)$, and a measure $\mu \in \Delta(\Pi)$, such that $\sum_{\pi \in \Pi} \phi \left(\sum_{s \in S} \pi(s) u(x_t(s)) \right) \mu(\pi) \geq \sum_{\pi \in \Pi} \phi \left(\sum_{s \in S} \pi(s) u(x(s)) \right) \mu(\pi)$ for all $x \in B(x_t, p_t)$, for all $t \in T$.

Theorem 2 *Let $(x_t, p_t)_{t \in T}$ be the data set. The following statements are equivalent:*

1. *The data set $(x_t, p_t)_{t \in T}$ is consistent with with the maximization of smooth preferences.*

2. *There exist a non-empty finite set N, $\Pi := \{\pi_n : n \in N\} \subset \Delta(S)$, $\mu \in \Delta(\Pi)$, and $(u_t, \phi_t, \rho_t, \lambda_t)_{t \in T}$, with $(u_t, \phi_t, \rho_t, \lambda_t) \in \mathbb{R}^{|S|} \times \mathbb{R}^{|N|} \times \mathbb{R}^{|N|}_{++} \times \mathbb{R}_{++}$ for each $t \in T$, such that*

$$u_{t'}(s') \leq u_t(s) + \lambda_t \frac{p_t(s)}{\sum_{n \in N} \pi_n(s) \rho_t(n) \mu(n)} (x_{t'}(s') - x_t(s)), \tag{S.1}$$

for all $(s, s', t, t') \in S \times S \times T \times T$, and

$$\phi_{t'}(n') \leq \phi_t(n) + \rho_t(n) \sum_{s \in S} (\pi_{n'}(s) u_{t'}(s) - \pi_n(s) u_t(s)), \tag{S.2}$$

for all $(n, n', t, t') \in N \times N \times T \times T$.

The intuition behind Theorem 2 is again simple. If the data set $(x_t, p_t)_{t \in T}$ is consistent with the maximization of smooth preferences, then x_t is a solution to the problem $\max_{x \in B(p_t, x_t)} \sum_{\pi \in \Pi} \phi\left(\sum_{s \in S} \pi(s) u(x(s))\right) \mu(\pi)$, and consequently (S.1) results from standard optimality conditions for convex problems. As for (S.2), it simply results from the concavity of ϕ, with the interpretation that $\rho_t(n)$ is the "derivative" of ϕ at $\sum_s u(x_t(s)) \pi_n(s)$, the expected utility of x_t given the belief π_n. Conversely, if the conditions (S.1) and (S.2) are satisfied, we construct increasing, continuous, and concave functions u and ϕ such that (u, ϕ, Π, μ) rationalize the data, as in Afriat (1967).

Note that if we define the probability π_t as

$$\pi_t(s) := \frac{\sum_{n \in N} \pi_n(s) \rho_t(n) \mu(n)}{\sum_{s \in S} \sum_{n \in N} \pi_n(s) \rho_t(n) \mu(n)},$$

for all $(s, t) \in S \times T$, and for all $t \in T$, let

$$\lambda_t^* = \frac{\lambda_t}{\sum_{s \in S} \sum_{n \in N} \pi_n(s) \rho_t(n) \mu(n)},$$

then (S.1) is identical to (V.1), so that the observational differences between smooth ambiguity and variational preferences are given solely by the conditions (V.2) and (S.2). Moreover, if the parameters ρ_t are independent of t (and thus the parameters π_t defined above are independent of t), then (V.2) and (S.2) hold trivially. This special case corresponds to the maximization of Savage preferences.

In general, suppose that we have found parameters that satisfy (S.1) (and hence (V.1)) and (S.2). This implies that there exist $(\phi_t)_t$ such that $\phi_{t'} \leq \phi_t + \chi_t \pi_t (u_{t'} -$

u_t) for all (t, t').[11] From Green and Srivastava (1986), a necessary and sufficient condition for the existence of such $(\phi_t)_t$ is that for any sequence $(t_0 = t, \ldots, t_{n+1} = t)$, $\sum_{i=1}^{n+1} \chi_{t_{i+1}} \pi_{t_{i+1}} (u_{t_i} - u_{t_{i+1}}) \geq 0$. Now, to satisfy (V.2) requires that we find $(c_t)_t$ such that $c_{t'} \geq c_t - (\pi_{t'} - \pi_t) u_t$ for all (t, t'). Again, from Green and Srivastava, this is equivalent to $\sum_{i=0}^{n} (\pi_{t_{i+1}} - \pi_{t_i}) u_{t_i} \geq 0$ for all sequences $(t_0 = t, \ldots, t_{n+1} = t)$. Noting that $\sum_{i=0}^{n} (\pi_{t_{i+1}} - \pi_{t_i}) u_{t_i} \geq 0$ is equivalent to $\sum_{i=1}^{n+1} \pi_{t_{i+1}} (u_{t_i} - u_{t_{i+1}}) \geq 0$, it follows that if we find parameters that satisfy (S.1) and (S.2) with the additional property that $\sum_{i=1}^{n+1} (\chi_{t_{i+1}} - 1) \pi_{t_{i+1}} (u_{t_i} - u_{t_{i+1}}) \leq 0$ for all sequences $(t_0 = t, \ldots, t_{n+1} = t)$, then we have found parameters that satisfy (V.1) and (V.2). Therefore, we have sufficient conditions for a data set consistent with the maximization of smooth preferences to also be consistent with the maximization of variational preferences.

3. Implementation

This section provides a small implementation of our tests using laboratory data from a portfolio choice experiment.[12] The primary objective is to demonstrate the revealed preference approach using both real and simulated data. A secondary objective is to contribute to the empirical literature on decision making under uncertainty, and in particular, to explore the observable implications of ambiguity aversion (see, for example, Ahn *et al.* (2011) and Hey and Pace (2012) for earlier empirical contributions in this direction).

3.1 Experimental Design

The experiment took place at the Adelaide Laboratory for Experimental Economics at the University of Adelaide. Seventy one subjects participated in the experiment, each completing ten decision problems. There were three states of the world and three Arrow-Debreu securities, one for each state. The likelihood of each state was fixed across problems, and this was emphasized to the subjects. In each decision problem, a subject was given a budget to purchase Arrow-Debreu securities. It was not possible to transfer resources across problems,[13] nor was any new information

[11] To see this, define $\chi_t := \sum_{s \in S} \sum_{n \in N} \pi_n(s) \rho_t(n) \mu(n)$, and $\phi_t := \sum_n \phi_t(n) \mu(n)$, multiply inequalities (S.2) by μ_n and sum over n.

[12] For complete details related to the implementation, which include the instructions given to subjects, a typical live screenshot, and all data and programs, see the supplementary online appendix available at http://www2.le.ac.uk/departments/economics/people/mpolisson.

[13] In fact, the interface required subjects to bind their budgets in each problem.

Problem #	1	2	3	4	5	6	7	8	9	10
Price Asset 1	15	30	20	10	25	10	40	20	30	45
Price Asset 2	15	25	15	20	25	20	30	25	25	40
Price Asset 3	20	40	25	15	35	20	30	30	35	45
Budget	325	350	450	450	500	425	325	350	300	325

Table 1: Decision Problems

provided from one decision to the next. Lastly, subjects were paid according to their choices across all decision problems.

Compared with previous experiments, our design had two distinctive features. Firstly, unlike previous designs inspired by Ellsberg's thought experiment, we did not provide the subjects with any prior information about the likelihoods of the states. In particular, we did not fix the probability of any state. Secondly, subjects were paid according to their choices across *all* decision problems. This contrasts with previous experiments in which a problem has been chosen at random and subjects have been paid according to the choices made in that particular problem.[14]

3.2 Data Analysis

Each subject in the experiment faced the same set of ten problems. The prices of the Arrow-Debreu securities and the budgets facing each subject are shown in Table 1. The average quantities of assets purchased across the sample of 71 subjects are shown in Table 2. Roughly speaking, the higher the price of an asset, the lower is the quantity purchased. However, these averages mask a great deal of heterogeneity. For example, across all ten choice problems, Subject 38 consumed only the second asset, Subject 41 equalized quantities purchased across assets, and Subject 42 equalized expenditures across assets.

As previously mentioned, in order to test whether a given data set satisfies the restrictions derived in the previous section requires that we make use of nonlinear programming techniques, which are known to be computationally hard. To circumvent these difficulties, we simplify each problem to a linear program by performing a grid search over the space of variables that enter the inequalities nonlinearly.

[14] Kuzmics (2012) proves that in such settings one cannot distinguish between subjective expected utility and the other models of interest.

Problem #	1	2	3	4	5	6	7	8	9	10
Quantity Asset 1	7.5	3.7	6.3	16.1	7.1	16.4	2.1	7.0	2.9	1.9
Quantity Asset 2	7.6	5.9	12.1	7.3	7.7	6.6	4.2	4.5	5.3	3.3
Quantity Asset 3	5.0	2.3	5.6	9.6	3.7	6.4	3.8	3.2	2.3	2.4

Table 2: Average Quantities

More precisely, the algorithm we adopt is as follows. Firstly, we fix a finite grid $\Pi \subset \Delta(S)$ of beliefs over the states. Secondly, for each $t \in T$, we fix $\pi_t \in \Pi$. Thirdly, for the fixed $(\pi_t)_{t\in T}$, we test for the existence of $(u_t, c_t, \lambda_t)_{t\in T}$ such that $(u_t, \pi_t, c_t, \lambda_t)_{t\in T}$ satisfies the conditions $(V.1)$ and $(V.2)$. If a feasible solution $(u_t, c_t, \lambda_t)_{t\in T}$ is found, the algorithm stops. If no feasible solution is found, we consider another vector of $(\pi_t)_{t\in T}$. The process is repeated until either a solution is found or all choices of $(\pi_t)_{t\in T} \in \Pi^{|T|}$ have been considered.

There are several remarks worth making. Firstly, for a fixed vector of $(\pi_t)_{t\in T}$, the conditions $(V.1)$ and $(V.2)$ are linear in the variables $(u_t, c_t, \lambda_t)_{t\in T}$. The techniques for solving linear programming problems are well established and computationally efficient. Secondly, the maximal number of loops (equivalently, the number of linear programs to solve) is $|\Pi|^{|T|}$. For instance, if the grid Π has 10 points (a relatively small grid), the maximal number of loops is already 10^{10}. To get a sense of the computational burden, notice that each linear program has 1000 inequalities and 50 unknowns.[15] It takes ALICE, the high performance computing facility at the University of Leicester, about 7 minutes to solve 10^5 linear programs of this size. To solve 10^{10} linear programs therefore takes about 486 days, and this is for a single observation $(x_t, p_t)_t$. However, we have a sample of 71 subjects (hence, 71 observations) and need to simulate thousands of artificial observations to compute the power of our tests. While we have been able to make considerable use of parallel computing, processing time was still a major issue; this led us to focus on sub-samples of the data, as we will see shortly. Thirdly, if we want to consider special cases of variational preferences, we must impose additional restrictions. For instance, for subjective expected utility, we need to impose that $\pi_t = \pi$ for each t. Fourthly, the algorithm we adopt establishes lower bounds on consistency. In other words, if the algorithm fails to find

[15] Since there are 3 states and 10 periods, there are $(3\times 10)^2$ inequalities (V.1) and 10^2 inequalities (V.2), and there are 50 unknowns: 30 $u_t(s)$, 10 λ_t, and 10 c_t. (Recall that the π_t's are fixed.)

a feasible solution, a feasible solution might still exist. An open issue is to design a more efficient algorithm.

A first look at the results in Table 3 shows that across the ten decision problems, 66 out of 71 subjects, or about 93 percent, satisfy GARP and are therefore consistent with some form of utility maximization. We now explore the consistency with maximization of Savage preferences (SEU).

With a grid width of 1/450, we find that 7 out of 71 subjects, or about 10 percent, are consistent with subjective expected utility.[16] To gauge success, it is often useful to draw a comparison with a natural alternative hypothesis of random choice, a convention established by Bronars (1987) and Beatty and Crawford (2011). As shown in Table 3, among 100,000 simulated data sets, we observe 76,356 that are consistent with some form of utility maximization and none that are consistent with subjective expected utility.[17] In other words, the probability of passing GARP when choosing randomly uniformly is approximately 0.76, compared to zero for subjective expected utility.[18] This suggests that subjective expected utility is highly restrictive across ten decision problems, and therefore it is perhaps a remarkable result that as many as 7 subjects satisfy these restrictions. Nevertheless, we argue that predictive success is greater in the more general model since it outperforms randomness to a larger extent.[19]

We now turn our attention to multiple prior preferences (MEU) and, more generally, to variational preferences (VAR). As previously explained, our strategy involves fixing a grid Π and then performing a grid search over $\Pi^{|T|}$ to check whether the conditions $(V.1)$ and $(V.2)$ are satisfied. Ideally, we would like the grid to be as fine as possible and to consider the entire sample. However, due to the high computational burden, we had to trade off between the width of the grid and the size of the subsets

[16] The grid width of 1/450 corresponds to a grid containing 100,576 elements, including the belief that all states are equally likely. Therefore, to implement the test for the maximization of Savage preferences, the algorithm may have to solve up to 100,576 linear systems, each containing 900 inequalities and 40 unknowns.

[17] To simulate a data set, we draw portfolios uniformly at random from the budget planes.

[18] From the Azuma-Hoeffding inequality for martingales, it follows that with 100,000 random draws, the empirical estimates are within 0.01 of their true values with probability at least 0.98.

[19] Selten (1991) and Beatty and Crawford (2011) axiomatize a measure of predictive success for set theories which predict a subset of the feasible outcome space. This measure satisfies natural monotonicity and equivalence properties, and furthermore, it is aggregable across a heterogeneous sample. This means that in order to estimate predictive success, we can take the simple difference between the sample pass rate and the estimated probability of consistency when choosing randomly uniformly.

	10 Problems		5 Problems				2 Problems			
	GARP	SEU	GARP	VAR	MEU	SEU	GARP	VAR	MEU	SEU
Pass	0.93	0.10	0.93	0.25	0.23	0.23	0.97	0.89	0.85	0.75
Power	0.76	0.00	0.78	0.00	0.00	0.00	0.86	0.38	0.27	0.24
Success	0.17	0.10	0.15	0.25	0.23	0.23	0.11	0.50	0.58	0.51
Width	1/450		1/6				1/27			

Table 3: Results

of data. We have chosen to explore two options: sub-samples of five observations with a grid width of 1/6 (i.e., 10 beliefs in Π), and sub-samples of two observations with a grid width of 1/27 (i.e., 325 beliefs in Π). Sub-samples are chosen from among the set of all sub-samples of a given size in order to minimize the probability of passing GARP when choosing randomly, i.e., to maximize the power of the test in a Bronars (1987) sense.

As Table 3 shows, with a grid width of 1/6 and across five decision problems, 66 out of 71 subjects satisfy GARP, again about 93 percent of the sample, compared to 77,606 out of 100,000 simulated data sets, or about 78 percent. The results for subjective and maxmin expected utility are completely identical, with 16 out of 71 subjects, or about 23 percent, satisfying both sets of restrictions, compared to a negligible 6 out 100,000 simulated data sets. Lastly, 18 out of 71 subjects, or about 25 percent, are consistent with variational preferences, compared to a meager 7 out of 100,000 simulated data sets. Our measure of predictive success suggests that variational preferences provide the best fit for the observed data, though only marginally better than maxmin or subjective expected utility, which are indistinguishable from one another. This exercise shows yet again that across only five choice problems, the structure we impose on preferences is very demanding.

Lastly, with a grid width of 1/27 and across two decision problems, 69 out of 71 subjects, about 97 percent, are consistent with GARP, compared to 86,149 out of 100,000 simulated data sets, about 86 percent; 53 out of 71 subjects, about 75 percent, are consistent with subjective expected utility, compared to 24,033 out of 100,000 simulated data sets, about 24 percent; 60 out of 71 subjects, about 85 percent, are consistent with maxmin expected utility, compared to 26,561 out of 100,000 simulated data sets, about 27 percent; and 63 out of 71 subjects, about 89 percent, are consistent with variational preferences, compared to 38,318 out of 100,000 simulated data sets, about 50 percent. By our measure of predictive success, maxmin expected utility performs the best, followed by subjective expected utility and then variational preferences. All of these models outperform the classical model of preference maximization (à la Debreu).

To conclude, testing whether a data set is consistent with the maximization of variational, multiple prior, or even Savage preferences has proven to be computationally challenging. We have not even been able to implement the tests for the model of smooth ambiguity due to the size and complexity of the space over which we must perform a grid search. An open issue is to develop better (both more efficient, and ideally, exact) algorithms to test these theories.

4. Concluding Remarks

This paper provides necessary and sufficient conditions for data sets composed of state-contingent prices and consumption to be consistent with two prominent models of decision making under uncertainty: variational preferences and smooth ambiguity. The revealed preference conditions for subjective expected utility, maxmin expected utility, and multiplier preferences are characterized as special cases. We implement our tests on data from a portfolio choice experiment, and we find evidence that favors making allowances for ambiguity, though subjective expected utility performs better than one might expect. We conclude with a series of brief remarks.

Concavity. We have assumed throughout that the Bernoulli utility function is concave. This is not without loss of generality. To see this, consider again the introductory example, but assume now that $B := \{(x_1, x_2, x_3) : x_1 + (4/5)x_2 + x_3 \leq 1\}$, i.e., the price of the consumption good in state s_3 has changed to 1. As in the introduction, the data set is inconsistent with a model of subjective expected utility. We further argue that it is also inconsistent with a model of maxmin expected utility if the utility function u is assumed to be concave. Without loss of generality, we normalize $u(0)$ to 0. Recall that since $(1, 0, 0)$ is chosen from B, we must have that $(1/3)u(1) \geq \min_{\pi \in \Pi}(1/3)u(x_1) + (2/3 - \pi)u(x_2) + \pi u(x_3)$ for any (x_1, x_2, x_3) such that $x_1 + (4/5)x_2 + x_3 = 1$, for some closed interval $\Pi \subseteq [0, 2/3]$. By monotonicity and concavity of u, for $(5/14, 5/14, 5/14)$, we have that $u(5/14) > u(1/3) \geq (1/3)u(1)$, and we obtain a contradiction. However, the data set is consistent with maxmin expected utility if we relax the concavity assumption. To see this, consider the piecewise linear utility function $u(x) = (9/10)x$ if $x \leq 5/9$ and $u(x) = (9/8)x - 1/8$ if $x > 5/9$. It is straightforward to verify, using linear programming techniques, that $(1, 0, 0)$ is a solution to the maximization problem $\max_{x\in B} \min_{\pi\in\Pi}(1/3)u(x_1) + (2/3 - \pi)u(x_2) + \pi u(x_3)$, while $(0, 1, 1)$ is a solution to the maximization problem $\max_{x\in B'} \min_{\pi\in\Pi}(1/3)u(x_1) + (2/3 - \pi)u(x_2) + \pi u(x_3)$. An open issue is to derive necessary and sufficient conditions without specific assumptions on the Bernoulli utility function u (beyond monotonicity, of course).

Errors. The conditions (V.1) and (V.2) and (S.1) and (S.2) are *exact* tests and they do not allow for measurement errors, optimization errors, or computational limitations.[20] If a test fails, a simple method to evaluate the seriousness of the violations consists of finding the largest subsets of the data that are consistent and comparing their cardinality with the cardinality of the data set. Another method, suggested by Varian (1985), is to assume that the observed data set $(x_t, p_t)_{t\in T}$ is the true data set $(x_t^*, p_t^*)_{t\in T}$ perturbed by classical error terms $(\varepsilon_{x_t}, \varepsilon_{p_t})_{t\in T}$, i.e., $x_t^* = x_t + \varepsilon_{x_t}$ and $p_t^* = p_t + \varepsilon_{p_t}$ for each $t \in T$, and to minimize

$$\frac{1}{\sigma^2} \sum_{t\in T} ||\hat{x}_t - x_t||^2 + ||\hat{p}_t - p_t||^2,$$

subject to $(\hat{x}_t, \hat{p}_t)_{t\in T}$ satisfying conditions (V.1) and (V.2) (resp., (S.1) and (S.2)), where σ^2 is the variance of the error terms.[21] If the true data set $(x_t^*, p_t^*)_{t\in T}$ is consistent with (V.1) and (V.2) (resp., (S.1) and (S.2)), i.e., the null hypothesis, then the resulting value of the minimization problem is a lower bound for the 'true' statistic $(1/\sigma^2)\sum_{t\in T} ||x_t^* - x_t||^2 + ||p_t^* - p_t||^2$, and therefore provides a conservative test.[22]

General Choice Sets. In this paper, the decision maker chooses from classical (linear) budget sets. However, the analysis is not limited to budget sets and extends to more general choice sets. More precisely, suppose that the data set consists of $(x_t, X_t)_{t\in T}$, where $x_t \in X_t$, and X_t is a non-empty and convex subset of $\mathbb{R}^\ell$ for each t. From standard arguments in convex analysis, condition (V.1) becomes

$$u_{t'}(s') \leq u_t(s) + g_t(x_t)(x_{t'}(s') - x_t(s)),$$

[20] See Echenique, Golovin, and Wierman (2011) for a revealed preference approach to computational complexity.

[21] Here, $||\cdot||$ denotes the Euclidean norm.

[22] Since we assume classical error terms, the statistic has a chi-squared distribution.

with $g_t(x_t)$ an element of the normal cone of X_t at x_t, while (V.2) remains the same. We refer the reader to Forges and Minelli (2009) for more on this issue.

Axioms of Revealed Preference. This paper derives Afriat inequalities for a data set to be consistent with variational preferences and smooth ambiguity. However, these inequalities are not stated purely in terms of observables, i.e., state-contingent prices and consumption. An open issue is to find testable implications that only involve observables, as in the generalized axiom of revealed preference.

Appendix

A.1 Proof of Theorem 1

Proof: We first show that (1) $\implies$ (2). Suppose that the data set $(x_t, p_t)_{t\in T}$ is consistent with a model of variational preferences. Since u is increasing, this is equivalent to x_t being a solution to the following constrained optimization program: $\max_{x\in\mathbb{X}} \left(\min_{\pi\in\Delta(S)} \left(\sum_{s\in S} \pi(s)u(x(s)) + c(\pi)\right)\right)$ subject to $p_t \cdot x \leq \omega_t$ for some $\omega_t \geq 0$ for each $t \in T$ (choose $\omega_t = p_t \cdot x_t$). For any $x \in \mathbb{X}$, let $U(x)$ denote the minimum over $\pi \in \Delta(S)$ of $\sum_{s\in S} \pi(s)u(x(s))+c(\pi)$, i.e., $U(x) := \min_{\pi\in\Delta(S)} \left(\sum_{s\in S} \pi(s)u(x(s)) + c(\pi)\right)$, and let $\Pi^{\min}(x)$ denote the set of minimizers. Notice that $\Pi^{\min}(x) \neq \emptyset$ for all $x \in \mathbb{X}$ and that U is concave. Note that the maximization program is equivalent to $\max_{x\in\mathbb{X}} U(x) - \mathbf{1}_{B(p_t,x_t)}(x)$, where $\mathbf{1}_{B(p_t,x_t)}$ is the indicator function of the non-empty, convex and closed set $B(p_t, x_t)$. Let $\partial U(x)$ be the super-differential of U at x. From the optimality of x_t, it follows that there exist a scalar $\lambda_t > 0$ and a vector $\delta_t \in \mathbb{R}_+^{\ell|S|}$ such that $\lambda_t p_t - \delta_t \in \partial U(x_t)$ for each $t \in T$, with $\delta_t^l(s) = 0$ if $x_t^l(s) > 0$ (i.e., if the quantity consumed of the l-th good in state s is strictly positive).

For each $s \in S$, define $u_s : \mathbb{X} \to \mathbb{R}$ with $u_s(x) = u(x(s))$. (Remember that $u : X \to \mathbb{R}$.) We have that $U(x) = \sum_{s\in S} u_s(x)\pi(s) + c(\pi)$ with $\pi \in \Pi^{\min}(x)$. From Theorem 4.4.2 in Hiriart-Urruty and Lemaréchal (2004, p. 189), we have that

$$\partial U(x) = \mathrm{co}\left\{\sum_{s\in S} \pi(s) \cdot g_s(x) : g_s(x) \in \partial u_s(x), \pi \in \Pi^{\min}(x)\right\},$$

where $\partial u_s(x)$ denotes the super-differential of u_s at x. Note that each element of the super-differential of u_s at x is a $\ell|S|$-dimensional vector. It follows that $\lambda_t p_t(s) - \delta_t(s) \in \{\pi_t(s) \cdot g(x_t(s)) : g(x_t(s)) \in \partial u(x_t(s)), \pi_t \in \Pi^{\min}(x_t)\}$. Concavity of u then implies that

$$u(x_{t'}(s')) \leq u(x_t(s)) + \lambda_t \frac{p_t(s)}{\pi_t(s)}(x_{t'}(s') - x_t(s)),$$

for all $(s, s', t, t') \in S \times S \times T \times T$, with π_t an element of $\Pi^{\min}(x_t)$ at x_t. Lastly, since π_t is an element of $\Pi^{\min}(x_t)$ at x_t, we have that $\sum_{s\in S} \pi_t(s)u(x_t(s)) + c(\pi_t) \leq \sum_{s\in S} \pi_{t'}(s)u(x_t(s)) + c(\pi_{t'})$ for all $(t, t') \in T \times T$. Letting $u_t(s) := u(x_t(s))$ for any $(s, t) \in S \times T$ and $c_t := c(\pi_t)$ for any $t \in T$ completes the first part of the proof.

We now show that (2) $\implies$ (1). Assume that there exist $(u_t, \pi_t, c_t, \lambda_t)_{t\in T} \in \mathbb{R}^{|S|} \times \Delta(S) \times \mathbb{R}_+ \times \mathbb{R}_{++}$, such that (V.1) and (V.2) are satisfied. Define $u : X \to \mathbb{R}$ as follows:

$$u(x) := \min_{(s,t)\in S\times T} \left(u_t(s) + \lambda_t \frac{p_t(s)}{\pi_t(s)}(x - x_t(s)) \right). \tag{1}$$

Note that u is increasing and concave. Also, we have that $u(x_t(s)) = u_t(s)$ for each $(s, t) \in S \times T$. Clearly, we have that $u(x_t(s)) \leq u_t(s)$. Assume that $u_t(s) > u(x_t(s))$. Since $u(x_t(s)) = u_{t^*}(s^*) + \lambda_{t^*} \frac{p_{t^*}(s^*)}{\pi_{t^*}(s^*)}(x_t(s) - x_{t^*}(s^*))$ for some $(s^*, t^*) \in S \times T$, we have a contradiction with (V.1). Similarly, define $c : \Delta(S) \to \mathbb{R}$ as follows:

$$c(\pi) := \max_{t\in T} \left(c_t - \sum_{s\in S} u_t(s)(\pi(s) - \pi_t(s)) \right). \tag{2}$$

Note that c is convex and continuous. Also, we have that $c(\pi_t) = c_t$ for each $t \in T$. Clearly, we have that $c(\pi_t) \geq c_t$. Assume that $c(\pi_t) > c_t$. Since $c(\pi_t) = c_{t^*} - \sum_{s\in S} u_{t^*}(s)(\pi_t(s) - \pi_{t^*}(s))$ for some $t^* \in T$, we have a contradiction with (V.2). To ground c, subtract $\min_{\pi\in\Delta(S)} c(\pi)$, which is well-defined by continuity of c and compactness of $\Delta(S)$. With a slight abuse of notation, we also denote by c the grounded function.

Finally, we want to show that for each $t \in T$, if $p_t \cdot x_t \geq p_t \cdot x$, then

$$\min_{\pi\in\Delta(S)} \left(\sum_{s\in S} \pi(s)u(x_t(s)) + c(\pi) \right) \geq \min_{\pi\in\Delta(S)} \left(\sum_{s\in S} \pi(s)u(x(s)) + c(\pi) \right).$$

We have that for each $t \in T$,

$$\begin{aligned}
\min_{\pi\in\Delta(S)} \left(\sum_{s\in S} \pi(s)u(x(s)) + c(\pi) \right) &\leq \sum_{s\in S} \pi_t(s)u(x(s)) + c(\pi_t) && (3)\\
&\leq \sum_{s\in S} \pi_t(s)u_t(s) + c(\pi_t) \\
&\quad + \lambda_t \sum_{s\in S} p_t(s)(x(s) - x_t(s)) && (4)\\
&\leq \sum_{s\in S} \pi_t(s)u_t(s) + c(\pi_t) && (5)\\
&= \sum_{s\in S} \pi_t(s)u_t(s) + c_t && (6)
\end{aligned}$$

$$\leq \min_{\pi \in \Delta(S)} \left(\sum_{s \in S} \pi(s) u_t(s) + c(\pi) \right) \quad (7)$$

$$= \min_{\pi \in \Delta(S)} \left(\sum_{s \in S} \pi(s) u(x_t(s)) + c(\pi) \right). \quad (8)$$

Inequality (3) follows from minimization; inequality (4) follows from the definition of u in (1); inequality (5) follows from the assumption that for each $t \in T$, $p_t \cdot x_t \geq p_t \cdot x$, which is equivalent to $\sum_{s \in S} p_t(s)(x(s) - x_t(s)) \leq 0$, and $\lambda_t > 0$; equality (6) follows from the previous argument that $c(\pi_t) = c_t$ for each $t \in T$; inequality (7) follows since the definition of c in (2) implies that $\sum_{s \in S} \pi_t(s) u_t(s) + c_t \leq \sum_{s \in S} \pi(s) u_t(s) + c(\pi)$ for all $\pi \in \Delta(S)$ and $t \in T$; and equality (8) follows from the previous argument that $u_t(s) = u(x_t(s))$ for each $(s,t) \in S \times T$. This completes the proof. **QED**

A.2 Proof of Theorem 2

Proof: We first show that (1) $\implies$ (2). Suppose that the data set $(x_t, p_t)_{t \in T}$ is consistent with the smooth ambiguity model. Since u and ϕ are increasing, this is equivalent to x_t being a solution to the following constrained optimization program: $\max_{x \in \mathbb{X}} \left(\sum_{\pi \in \Pi} \phi \left(\sum_{s \in S} \pi(s) u(x(s)) \right) \mu(\pi) \right)$ subject to $p_t \cdot x \leq \omega_t$ for some $\omega_t \geq 0$ for each $t \in T$ (choose $\omega_t = p_t \cdot x_t$). Let $U_\pi(x) := \sum_{s \in S} \pi(s) u(x(s))$ for each $x \in \mathbb{X}$ and $\pi \in \Pi$. Since u is concave, U_π is also concave for each $\pi \in \Pi$. Since ϕ is concave, it follows that the super-differential of $\phi \circ U_\pi$ at x is given by $\{\rho \cdot g : \rho \in \partial\phi(U_\pi(x)), g \in \partial U_\pi(x)\}$. (See Theorem 4.3.1 in Hiriart-Urruty and Lemaréchal (2004, p. 186).) From the optimality of x_t, it follows that there exist $\lambda_t > 0$ for each $t \in T$, $g(x_t(s)) \in \partial u(x_t(s))$ for each $s \in S$ and $t \in T$, and $\rho_\pi(x_t) \in \partial\phi(U_\pi(x_t))$ for each $\pi \in \Pi$ and $t \in T$, such that $\lambda_t p_t(s) = g(x_t(s)) \sum_{\pi \in \Pi} \pi(s) \rho_\pi(x_t) \mu(\pi)$ for each $s \in S$ and $t \in T$. (Here, we implicitly assume that $x_t >> 0$. The general case is treated as in the proof of Theorem 1.) Concavity of u then implies that

$$u(x_{t'}(s')) \leq u(x_t(s)) + \lambda_t \frac{p_t(s)}{\sum_{\pi \in \Pi} \pi(s) \rho_\pi(x_t) \mu(\pi)} (x_{t'}(s') - x_t(s)),$$

for all $(s, s', t, t') \in S \times S \times T \times T$. Similarly, concavity of ϕ implies that

$$\phi(U_{\pi'}(x_{t'})) \leq \phi(U_\pi(x_t)) + \rho_\pi(x_t)(U_{\pi'}(x_{t'}) - U_\pi(x_t)),$$

for all $(n, n', t, t') \in N \times N \times T \times T$. Letting $u_t(s) := u(x_t(s))$ for any $(s,t) \in S \times T$, $\pi_n(s) := \pi(s)$ for any $(\pi, s) \in \Pi \times S$, $\phi_t(n) := \phi(U_\pi(x_t))$ and $\rho_t(n) := \rho_\pi(x_t)$ for any $(\pi, t) \in \Pi \times T$, and $\mu(n) := \mu(\pi)$ for any $\pi \in \Pi$ completes the first part of the proof.

Conversely, we next show that (2) $\implies$ (1). Assume that there exist $\Pi := \{\pi_n\}_{n\in N} \subset \Delta(S)$, $\mu \in \Delta(\Pi)$, and $(u_t, \phi_t, \rho_t, \lambda_t)_{t\in T} \in \mathbb{R}^{|S|} \times \mathbb{R}^{|N|} \times \mathbb{R}^{|N|}_{++} \times \mathbb{R}_{++}$, such that (S.1) and (S.2) are satisfied. Define $u : X \to \mathbb{R}$ as follows:

$$u(x) := \min_{(s,t)\in S\times T} \left(u_t(s) + \lambda_t \frac{p_t(s)}{\sum_{n\in N} \pi_n(s)\rho_t(n)\mu(n)} (x - x_t(s)) \right). \tag{9}$$

Note that u is increasing and concave. Also, we have that $u(x_t(s)) = u_t(s)$ for each $(s,t) \in S \times T$. Clearly, we have that $u(x_t(s)) \le u_t(s)$. Assume that $u_t(s) > u(x_t(s))$. Since $u(x_t(s)) = u_{t^*}(s^*) + \lambda_{t^*} \frac{p_{t^*}(s^*)}{\sum_{n\in N} \pi_n(s^*)\rho_{t^*}(n)\mu(n)} (x_t(s) - x_{t^*}(s^*))$ for some $(s^*, t^*) \in S \times T$, we have a contradiction with (S.1). Similarly, define $\phi : \mathbb{R} \to \mathbb{R}$ as follows:

$$\phi(U) := \min_{(n,t)\in N\times T} \left(\phi_t(n) + \rho_t(n) \left(U - \sum_{s\in S} \pi_n(s)u_t(s) \right) \right). \tag{10}$$

Note that ϕ is increasing and concave. Also, we have that $\phi\left(\sum_{s\in S} \pi_n(s)u_t(s)\right) = \phi_t(n)$ for each $(n,t) \in N \times T$. Clearly, we have that $\phi\left(\sum_{s\in S} \pi_n(s)u_t(s)\right) \le \phi_t(n)$. Assume that $\phi_t(n) > \phi\left(\sum_{s\in S} \pi_n(s)u_t(s)\right)$. Since $\phi\left(\sum_{s\in S} \pi_n(s)u_t(s)\right) = \phi_{t^*}(n^*) + \rho_{t^*}(n^*)\left(\sum_{s\in S}(\pi_n(s)u_t(s) - \pi_{n^*}(s)u_{t^*}(s))\right)$ for some $(n^*, t^*) \in N\times T$, we have a contradiction with (S.2). Finally, we want to show that for each $t \in T$, when $p_t \cdot x_t \ge p_t \cdot x$, it then follows that $\sum_{\pi\in\Pi} \phi\left(\sum_{s\in S} \pi(s)u(x_t(s))\right)\mu(\pi) \ge \sum_{\pi\in\Pi} \phi\left(\sum_{s\in S} \pi(s)u(x(s))\right)\mu(\pi)$. We have that for each $t \in T$,

$$u(x(s)) - u(x_t(s)) \le \lambda_t \frac{p_t(s)}{\sum_{n\in N} \pi_n(s)\rho_t(n)\mu(n)} (x(s) - x_t(s)), \tag{11}$$

which follows from the definition of u in (9) and the previous argument that $u(x_t(s)) = u_t(s)$ for each $(s,t) \in S \times T$. We have that for each $n \in N$ and $t \in T$,

$$\phi\left(\sum_{s\in S} \pi_n(s)u(x(s))\right) \le \phi\left(\sum_{s\in S} \pi_n(s)u(x_t(s))\right) + \rho_t(n)\sum_{s\in S} \pi_n(s)(u(x(s)) - u(x_t(s))), \tag{12}$$

which follows from the definition of ϕ in (10) and the previous arguments that $u(x_t(s)) = u_t(s)$ for each $(s,t) \in S \times T$ and $\phi\left(\sum_{s\in S} \pi_n(s)u_t(s)\right) = \phi_t(n)$ for each $(n,t) \in N \times T$. Together, (11) and (12) imply that

$$\phi\left(\sum_{s\in S} \pi_n(s)u(x(s))\right) \le \phi\left(\sum_{s\in S} \pi_n(s)u(x_t(s))\right) + \lambda_t \sum_{s\in S} \frac{\pi_n(s)\rho_t(n)}{\sum_{n\in N} \pi_n(s)\rho_t(n)\mu(n)} p_t(s)(x(s) - x_t(s)), \tag{13}$$

for each $n \in N$ and $t \in T$. Multiplying both sides of (13) by $\mu(n)$ and summing over $n \in N$, we obtain

$$\sum_{n\in N} \phi\left(\sum_{s\in S} \pi_n(s)u(x(s))\right)\mu(n) \leq \sum_{n\in N} \phi\left(\sum_{s\in S} \pi_n(s)u(x_t(s))\right)\mu(n) + \lambda_t \sum_{s\in S} p_t(s)(x(s) - x_t(s)). \tag{14}$$

Finally, if for each $t \in T$, $p_t{\cdot}x_t \geq p_t{\cdot}x$, which is equivalent to $\sum_{s\in S} p_t(s)(x(s)-x_t(s)) \leq 0$, then from (14) we have

$$\sum_{n\in N} \phi\left(\sum_{s\in S} \pi_n(s)u(x(s))\right)\mu(n) \leq \sum_{n\in N} \phi\left(\sum_{s\in S} \pi_n(s)u(x_t(s))\right)\mu(n). \tag{15}$$

Letting $\mu(\pi_n) := \mu(n)$ for each $n \in N$ and $\Pi := \{\pi_n\}_{n\in N}$ in (15) completes the proof. **QED**

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