THE UNIVERSITY OF ADELAIDE AUSTRALIA

The University of Adelaide
School of Economics

Research Paper No. 2013-15
October 2013

Number of Sellers and Quantal Response Equilibrium Prices

Ralph-C. Bayer and Hang Wu

Number of Sellers and Quantal Response Equilibrium Prices*

Ralph-C. Bayer† & Hang Wu‡

Abstract

This paper studies the effects of increasing the number of sellers on Quantal Response Equilibrium (QRE) prices in homogeneous product Bertrand oligopoly markets. We show that the two most commonly used choice functions (power and logistic) lead to qualitatively different comparative-static predictions with respect to the relationship between number of firms and prices.

JEL Numbers: C73, D83, L13

Keywords: Bertrand Oligopoly, Quantal Response Equilibrium, Comparative Statics

*The authors would like to acknowledge the financial support of the Australian Research Council under DP120101831.

†University of Adelaide, School of Economics, email: ralph.bayer@adelaide.edu.au.

‡Corresponding author: University of Adelaide, School of Economics, North Terrace, Adelaide, SA 5005, Australia, email: hang.wu@adelaide.edu.au, Tel.: ++61 (0)8 8313 6024.

1 Introduction

Several recent studies show that Quantal Response Equilibrium (cf. McKelvey & Palfrey 1995) can effectively explain price dispersion observed in laboratory Bertrand markets (e.g., Baye & Morgan 2004; Capra et al. 2002; and Dufwenberg et al. 2007). At a Quantal Response Equilibrium (QRE), sellers are assumed to be boundedly rational and play noisy best responses to their beliefs about other players' strategies. Moreover, the beliefs held by all sellers are correct and prices with higher expected payoffs are more likely to be played. This paper investigates how the changes in the number of sellers affects QRE market prices.

We study two different but closely related specifications of QRE, namely, the power-function specification and the logistic specification. In the literature, the two specifications are frequently used and considered as substitutes that lead to qualitatively similar equilibria. Under the power function specification (a generalization of Luce 1959), the probability of posting a price is proportional to a power function of its expected payoff. In contrast, for the more widely used logistic specification the choice probabilities of prices are proportional to the exponential of the corresponding expected payoffs (e.g., McFadden 1973). Since closed form solutions are not readily available for both specifications we propose a simple Cournot adjustment algorithm, which by design yields a QRE if it converges. We find that, quite surprisingly, an increase of the number of sellers has opposing effects on the average QRE price under the two specifications. Under the power-function specification, as expected an increase in the number of competing sellers results in a decrease of the average market price. In contrast, under the logistic specification, having more sellers increases the average market price.

The remainder of the paper is structured as follows. The next section lays out the two Quantal Response Equilibrium approaches of modeling price dispersion in Bertrand oligopoly markets. Section 3 presents the numerical solutions of the QRE models. Section 4 concludes.

2 Theory

Consider a Bertrand oligopoly market with n sellers engaged in price competition for selling one unit of a homogeneous product. All sellers produce at identical and constant marginal costs c which, without loss of generality, is normalized to zero. Let v be the choke-off price above which demand is zero. Sellers choose prices simultaneously and independently from the non-trivial price interval $\mathrm{P} \equiv [0, v]$. A seller whose price is lower than all other sellers' prices serves the market alone and earns the corresponding profit. If a seller is not among the lowest priced sellers, she does not sell and earns zero profit. In the case that m sellers are tied at the lowest price, they split the market equally. Given the price profile $(p_1, p_2 \cdots, p_n)$, for seller $i \in N \equiv \{1, 2, \cdots, n\}$ the payoff function can be formalized by

$$\Pi_i(p_1, p_{2,} \cdots, p_n) = \begin{cases} p_i & \text{if } p_i < p_j \forall j \neq i \\ \frac{p_i}{m} & \text{if } m \text{ sellers are tied at the lowest pirce } p_i \\ 0 & \text{otherwise} \end{cases}$$

The only symmetric Bertrand-Nash equilibrium for the game is for all sellers to set prices at zero. Quantal Response Equilibrium models bounded rationality by allowing sellers to choose suboptimal strategies with positive probability. The strategy of seller $i \in N$ is a probability measure over the price set P, i.e., $F_i : \mathrm{P} \longrightarrow [0, 1]$. Further, let $B_i(F_{-i})$ be seller i's belief about the probability measure of her opponents' strategies. Accordingly, for seller i the expected payoff of choosing price p is expressed as

$$E\pi_i\left(p, B(F_{-i})\right) = p\left[1 - B_i(F_{-i}(p))\right]^{n-1}. \tag{1}$$

Given the expected payoff function, the cumulative probability of seller i choosing price p, under the logistic specification, is stated as

$$F_i^{LS}\left(p, B_i(F_{-i})\right) = \frac{\int_0^p \mathrm{e}^{\lambda \cdot E\pi_i(q, B(F_{-i}))} dq}{\int_0^v \mathrm{e}^{\lambda \cdot E\pi_i(k, B(F_{-i}))} dk}, \forall p \in \mathrm{P}, \forall i \in N. \tag{2}$$

Replacing $E\pi_i\left(p, B(F_{-i})\right)$ in Equation 2 by $\ln\left[E\pi_i\left(q, B(F_{-i})\right)\right]$ yields the corresponding power function specification as

$$F_i^{PS}\left(p, B_i(F_{-i})\right) = \frac{\int_0^p E\pi_i\left(q, B(F_{-i})\right)^{\lambda} dq}{\int_0^v E\pi_i\left(k, B(F_{-i})\right)^{\lambda} dk}, \forall p \in \mathrm{P}, \forall i \in N. \tag{3}$$

Common to both specifications, the probability of a seller posting a price is positively related to the expected profit it yields. Better choices are played with higher probabilities but the best choice is not played with certainty. The parameter $\lambda \in [0, \infty)$ measures the degree of bounded rationality in sellers' decisions. For $\lambda = 0$, subjects randomize with identical probability over all prices. As λ increases, sellers become more precise in making choices, and as λ tends to infinity they become fully rational and choose the best response to their beliefs with probability one. At a QRE, all sellers' strategies are quantal responses to their beliefs about the competing sellers strategies and the beliefs are correct.

Definition. A strategy profile $F^{LQRE} \in [0, 1]^n$ comprises a Logistic Quantal Response Equilibrium *(LQRE),* if for all $i \in N$ and $p \in \mathrm{P}$, we have $F_i^{LQRE}(p) = F_i^{LS}(p_i, B_i(F_{-i}))$

and $B_i(F_{-i}(p)) = F_{-i}^{LQRE}(p)$. Similarly, a strategy profile $F^{PQRE} \in [0,1]^n$ comprises a Power-function Quantal Response Equilibrium *(PQRE),* if for all $i \in N$ and $p \in \mathrm{P}$, we have $F_i^{PQRE}(p) = F_i^{PS}(p_i, B_i(F_{-i}))$ and $B_i(F_{-i}(p)) = F_{-i}^{PQRE}(p)$.

For the power function specification, Baye & Morgan (2004) show that a closed-form representations of symmetric F^{PQRE} can be obtained and comparative static properties of $PQRE$ can be analytically studied.[1]

Proposition 1. *(Baye & Morgan 2004) For any $\lambda \in [0, \frac{1}{n-1})$, the following comprises a symmetric PQRE:*

$$F_i^{PQRE}(p) = 1 - \left[1 - \left(\frac{p}{v}\right)^{1+\lambda}\right]^{\frac{1}{1-(n-1)\lambda}} \quad \forall p \in \mathrm{P},\, \forall i \in I \tag{4}$$

Two features of Proposition 1 are noteworthy. First, for a given $\lambda \in [0, \frac{1}{n-1})$, $F_i^{PQRE}(p)$ is uniquely determined. When $\lambda = 0$, prices are uniformly distributed over the price set. As λ approaches $\frac{1}{n-1}$, $F_i^{PQRE}(p)$ tends to one for all $p \in \mathrm{P}$, which corresponds to the Bertrand-Nash equilibrium outcome. Second, equation (4) is only giving a valid equilibrium density for $\lambda < \frac{1}{n-1}$, as it would yield negative cumulative densities for a greater λ. For $\lambda \in [0, \frac{1}{n-1})$, an increase in n leads to a higher $F_i^{PQRE}(p)$ for all $p \in (0, v)$, and thus to a decrease in the PQRE price in terms of first-order stochastic dominance.

Unfortunately, there is no closed-form solution for the LQRE probability function, and thus we cannot study the comparative static properties of LQRE analytically. For this reason, in the next section we numerically determine equilibria with both specifications and investigate the effects of increasing seller numbers.

3 Numerical Solutions

For our numerical algorithm we consider a discretized version of the game with price set $\mathrm{P} \equiv \{0, 1, \cdots, 100\}$. We consider $n \in \{2, 3, 4, 5\}$ with $\lambda = 0.15$. We use $\lambda = 0.15$ because we know from Equation 4 that a symmetric QRE exists for up to $n = 6$ under the power specification. Rather than calculating the QRE probability distribution of prices

[1] For a detailed analysis and an experimental test of the comparative static properties of the PQRE, see Dufwenberg et al. (2007).

directly, we adopt a simple Cournot adjustment algorithm that leads to a QRE once it converges and can also be interpreted as describing learning dynamics. The Cournot processes start with a uniform probability measure of prices $F_{t=0}(p) = \frac{p+1}{101} \forall p \in \mathrm{P}$. For each of the following periods, $F_{t=1,2,3,\cdots}$ is defined as a quantal response to F_{t-1}. Formally, the Cournot process for the power function specification is given as

$$F_t^{PS}(p) = \frac{\sum_{q=0}^{p} E\pi\left(q, F_{t-1}^{PS}\right)^{\lambda}}{\sum_{k=0}^{v} E\pi\left(k, F_{t-1}^{PS}\right)^{\lambda}}, \forall p \in \mathrm{P}, \forall t = 1, 2, 3 \cdots ; \tag{5}$$

similarly, the process for the logistic specification is stated as

$$F_t^{LS}(p) = \frac{\sum_{q=0}^{p} e^{\lambda \cdot E\pi\left(q, F_{t-1}^{LS}\right)}}{\sum_{k=0}^{v} e^{\lambda \cdot E\pi\left(k, F_{t-1}^{LS}\right)}}, \forall p \in \mathrm{P}, \forall t = 1, 2, 3 \cdots . \tag{6}$$

If the Cournot process converges to a steady state, then the strategy profile in which all sellers adopting the corresponding mixed strategies at the steady state is necessarily a symmetric Quantal Response Equilibrium. To see this recall the two necessary conditions for a QRE. First, all sellers are playing quantal responses to the belief that other sellers will keep playing the same strategies. Second, all sellers' beliefs are consistent to the strategies played by their opponents. The first condition is clearly satisfied, since we induce quantal best responses. Convergence implies that the believed price distribution F_{t-1} is identical to the played price distribution F_t, which implies that the second condition is satisfied.

The Cournot process converges if

$$\lim_{t \to \infty} \max_{p \in \mathrm{P}} |f_t(p) - f_{t-1}(p)| = 0.$$

For each of the four cases, we simulate the Cournot process for a thousand periods. As can be seen from Figure 1, for both specifications the Cournot processes converge in all $n = 2, 3, 4, 5$ scenarios.

Insert Figure 1 about here.

For the power function specification, the results of the simulations are consistent with implications that can be derived from Proposition 1. Figure 2(A) shows the average price dynamics of the Cournot processes. As n increases from two to five, in equilibrium the average market price declines from 49.2 to 33.6. Figure 2(B) shows the corresponding probability distributions at the steady states. As n increases, in the PQRE increasingly more probability mass is shifted from the high price domain (50 to 100) to the low price domain (0 to 50).

Insert Figure 2 about here.

Now we turn to the simulations of the logistic specification. Surprisingly, increasing the number of sellers affects LQRE in the opposite direction than PQRE. As shown in Figure 3(A), the average LQRE market price increases from 37.8 in the duopoly scenario to 42.5 in the pentaopoly scenario. Figure 3(B) shows the corresponding probability distributions at the LQRE. In contrast to the PQRE, in LQRE as n increases, prices at the two ends of the price interval attract more probability mass, and those in the middle range are allocated lower mass. The increase in probabilities of the high prices is the main force that drives up the equilibrium average market price.

Insert Figure 3 about here.

So far we have focused on $\lambda < \frac{1}{n-1}$, where the condition of Proposition 1 is satisfied and the Cournot processes converge. It is also interesting to investigate how would a change in the number of sellers affect the Cournot dynamics if we relax the restriction and simulate the models with a higher λ. We conduct a new set of simulations using $\lambda = 0.4$ while keeping everything else unchanged. Figure 4 presents the results of simulations for the power-function specification (Panel A) and the logistic specification (Panel B). For the power-function specification, when n equals two or three (we still have $\lambda < \frac{1}{n-1}$ for these two cases), the Cournot processes still converge. However, when n equals three or four, the Cournot processes evolve cyclically after a few periods and the cycles persist over time.

Insert Figure 4 about here.

For the logistic specification, the simulations still produce results that contrast those obtained from the simulations with the power-function specification. In the duopoly, triopoly and quadropoly scenarios we observe persistent cycles of average market prices. However, increasing the number of sellers reduces the amplitudes of the price cycles and, when the number of sellers is increased to five the process converges after about 30 periods. We also ran simulations for markets with six to ten sellers. In all of these cases the Cournot process also converges, and the results indicate that as the number of sellers increases the speed of convergence increases. Again, increasing the number of competitors increases the market price. Therefore, we conclude that increasing the number of sellers may result in nonconvergence of the Cournot process under the power function specification, while it can lead to convergence of the process under the logistic specification.

4 Conclusion

The conventional viewpoint that increased competition among sellers has the effect of reducing market prices has been challenged by many authors. Some models introduce search costs for consumers hunting for the lowest market price (e.g., Satterthwaite 1979; Stiglitz 1987; and Janssen & Moraga-González 2004). With more sellers competing in the market it is more costly for the consumers to succeed when searching. This effect can reduce search intensity, which in turn gives firms more market power. The possible result are increased prices. An alternative approach is to divide the consumers into two different types, for instance, loyal and swinging buyers in Rosenthal (1980) or, informed and uninformed buyers in Varian (1980). When facing intensified competition the sellers may have an incentive to exploit the loyal buyers and the uninformed buyers by charging a higher price. This paper demonstrates that the same phenomenon can arise in homogeneous product Bertrand oligopoly markets with identical bounded rational sellers in a QRE. The results of this paper also indicate that caution is necessary when choosing the quantal response specification to model Bertrand competition. The two dominant specifications which typically are seen as substitutes in modeling, lead to qualitatively vastly different results with respect to the impact of number of firms on price levels.Which specification is appropriate as the mean prices effect of increased seller numbers remains an empirical question though.

References

Baye, M. R. & Morgan, J. (2004), 'Price dispersion in the lab and on the internet: Theory and evidence', *The RAND Journal of Economics* **35**(3), 449–466.

Capra, C. M., Goeree, J. K., Gomez, R. & Holt, C. A. (2002), 'Learning and noisy equilibrium behavior in an experimental study of imperfect price competition', *International Economic Review* **43**(3).

Dufwenberg, M., Gneezy, U., Goeree, J. & Nagel, R. (2007), 'Price floors and competition', *Economic Theory* **33**(1), 211–224.

Janssen, M. C. & Moraga-González, J. L. (2004), 'Strategic pricing, consumer search and the number of firms', *The Review of Economic Studies* **71**(4), 1089–1118.

Luce, R. (1959), *Individual choice behavior.*, John Wiley.

McFadden, D. (1973), 'Conditional logit analysis of qualitative choice behavior'.

McKelvey, R. D. & Palfrey, T. R. (1995), 'Quantal response equilibria for normal form games', *Games and Economic Behavior* **10**(1), 6–38.

Rosenthal, R. W. (1980), 'A model in which an increase in the number of sellers leads to a higher price', *Econometrica: Journal of the Econometric Society* pp. 1575–1579.

Satterthwaite, M. A. (1979), 'Consumer information, equilibrium industry price, and the number of sellers', *The Bell Journal of Economics* pp. 483–502.

Stiglitz, J. E. (1987), 'Competition and the number of firms in a market: Are duopolies more competitive than atomistic markets?', *The Journal of Political Economy* **95**(5), 1041–1061.

Varian, H. R. (1980), 'A model of sales', *The American Economic Review* **70**(4), 651–659.

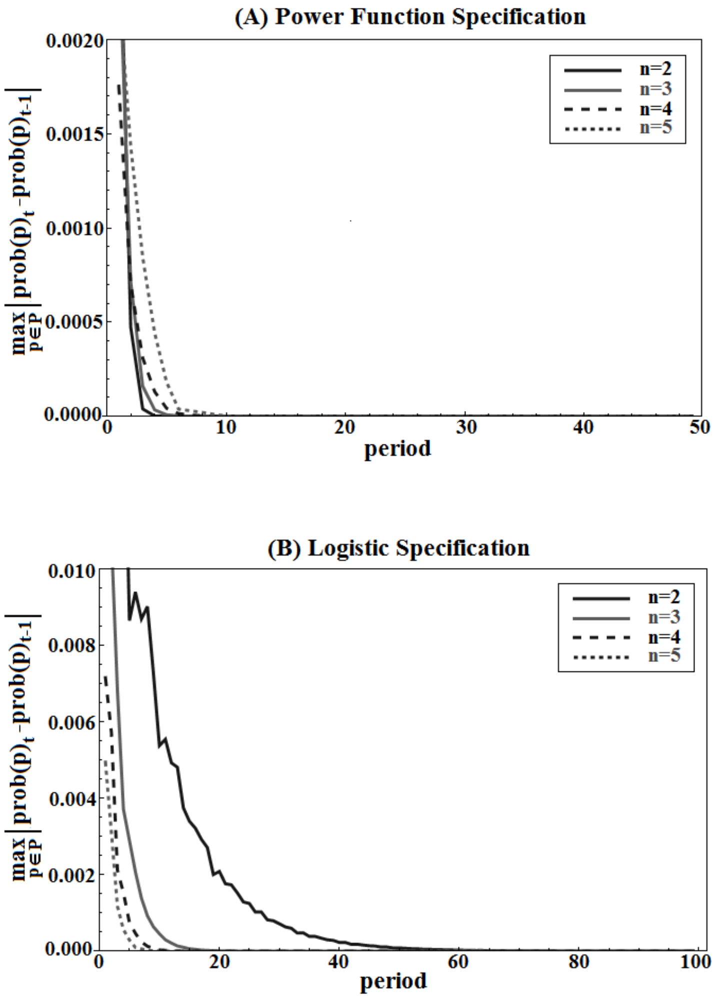

Figure 1: Time Series for the Maximum Absolute Value of Inter-temporal Changes in Price Densities with ($\lambda = 0.15$)

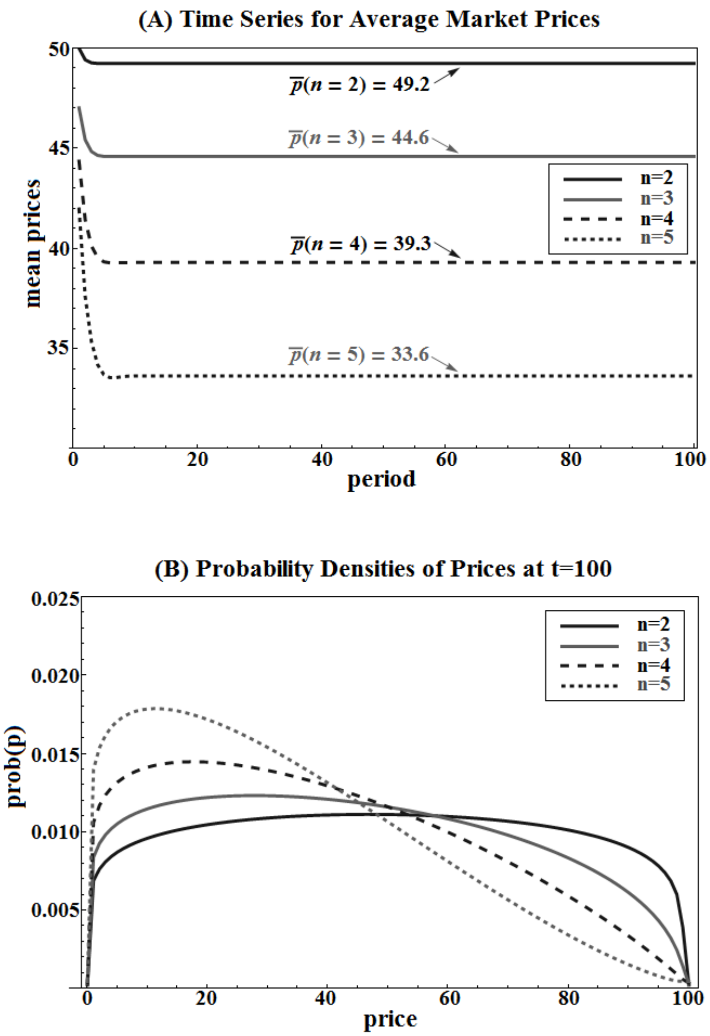

Figure 2: Time Series of Average Market Prices (A) and PQRE Probability Densities of Prices (B) : Power Function Specification with $\lambda = 0.15$

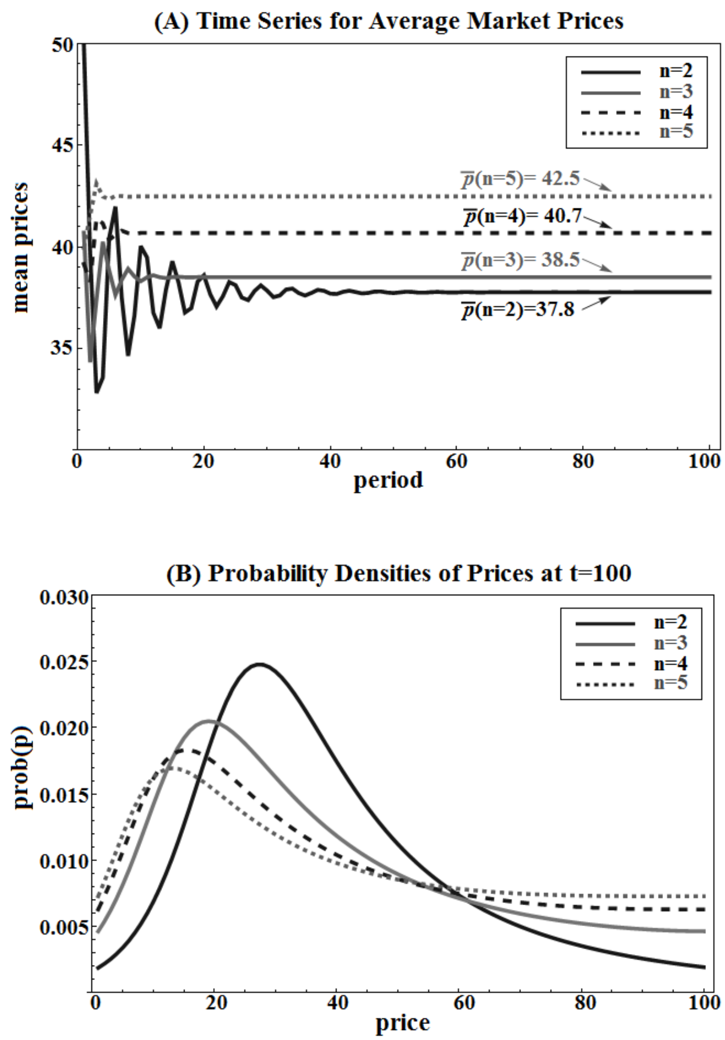

Figure 3: Time Series of Average Market Prices (A) and LQRE Probability Densities of Prices (B) : Logistic Specification with $\lambda = 0.15$

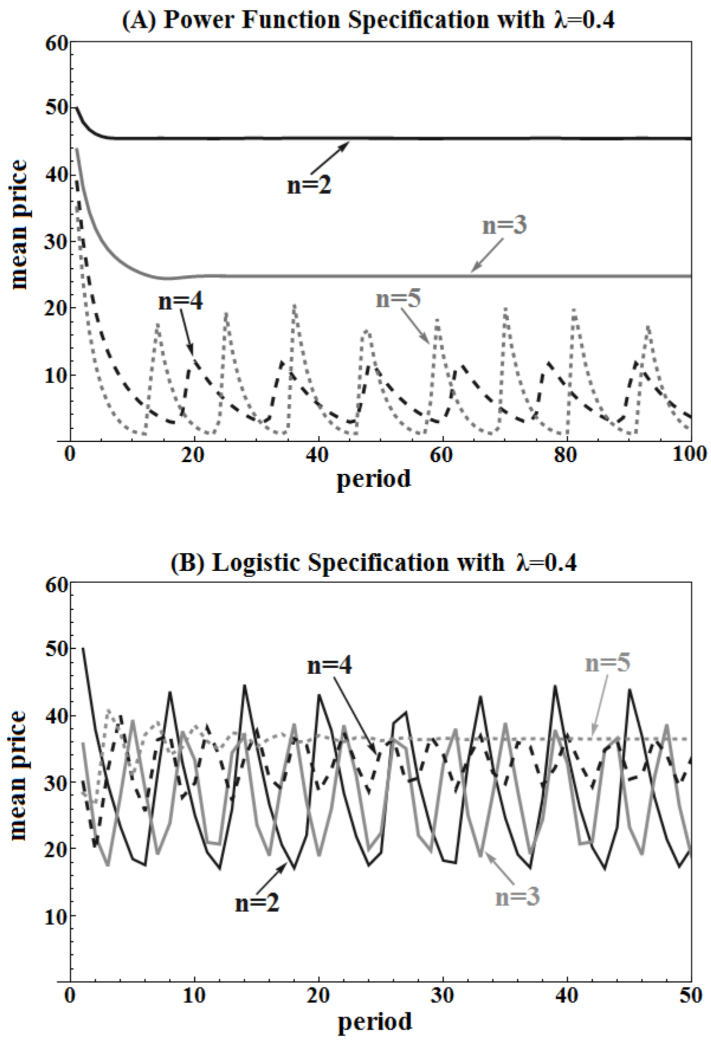

Figure 4: Time Series of Average Market Prices with $\lambda = 0.4$.