The University of Adelaide
School of Economics

Research Paper No. 2013-20
November 2013

Incentives for Research Agents: Optimal Contracts and Implementation

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Abstract

We study the agency problem between a firm and its research employees. In a multi-agent contracting setting, we show explicitly the way in which the optimal incentive regime is a function of how agents' efforts interact with one another: relative-performance evaluation is used when their efforts are substitutes whereas joint-performance evaluation is used when their efforts are complements. We also provide an implementation of the optimal contract, in which a primary component of the agents' compensation is a risky security. This implementation gives a theoretical justification for the wide-spread use of stock-based compensation by firms that rely on R&D.

Key words: Dynamic Contract, Repeated Moral Hazard, Multi-agent Incentive, R&D, Employee Compensation

JEL: D23, D82, D86, J33, L22, O32

*A special thanks to my advisor, Srihari Govindan, for his guidance and thoughtful support. I would like to thank Martin Gervais, Ayca Kaya, Kyungmin Kim, B. Ravikumar, Raymond Riezman, Yuzhe Zhang, and seminar participants at University of Iowa, University of Rochester, University of Adelaide, 2013 North American Summer Meeting of the Econometric Society in Los Angeles, 2013 Australasia Meeting of the Econometric Society in Sydney, and 2013 Asia Meeting of the Econometric Society in Singapore for their valuable advice and comments. All remaining errors are my own.

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1 Introduction

Over the last decade, the industries of information and communication technologies have become the engines of the U.S. economy: during this period, they have had an average share of 4.6 percent of gross GDP and have accounted for one-fourth of GDP-growth. A distinct feature of these so-called "new-economy" industries is a substantial investment in R&D, which constituted a third of business-sector expenditure on R&D in 2007 according to a recent report of the Bureau of Economic Analysis. Clearly, the success of firms in these industries depends crucially on the performance of the employees in their R&D units, and compensation schemes for these researchers become a major decision for these firms. This decision-problem shares some features with the standard problem of providing incentives to workers, but it also has some unique features. Like its standard counterpart, a moral-hazard phenomenon arises in this specific agency relationship. The outcome of research is uncertain, i.e. effort put into research today will not necessarily lead to a discovery tomorrow. However, the stochastic process governing the outcomes is influenced by how much effort is put into research: higher levels of effort increase the chance of making a discovery. Owing to task-complexity, effort exerted by researchers is difficult to monitor. Now, if the effort level is unobservable, then the imperfect monitoring of effort combined with the stochastic feature of innovation maps into a moral-hazard problem. Furthermore, since most R&D projects last a long period of time, the moral-hazard problem is dynamic in nature.

The agency problem that these R&D-intensive firms face with respect to their research employees differs from the standard principal-agent problems studied in literature in two aspects. First, some R&D projects progress through different phases, with research in each phase depending on the outcomes of previous phases. In these new-economy industries, this feature is particularly prominent. For example, Microsoft has released a sequence of Windows operating systems since 1985, from Windows 3.0, to Windows XP, and then to the most recent version, Windows 8. In each upgrade, Microsoft introduced a number of new features which make the use of computers easier and more convenient.

The second point of departure from standard agency problems is that R&D projects are nowadays typically undertaken by groups of researchers. Unlike the era when Edison invented the light bulb, and Bell telephone, R&D projects are now so complicated that great technological breakthroughs are seldom obtained by individual effort. Large efficiencies can be achieved when multiple researchers target the same hurdle in technological development. Hence, the most innovative companies in the world, like Apple, Google, Microsoft, IBM, and Sony, have adopted innovation-teams, which enable them to launch innovations faster. The wide spread use of team in R&D projects suggests that a multilateral environment is the appropriate setting to think about the agency problem between a firm and its in-house R&D unit.

The agency problem faced by these new-economy firms combines the two features described above, namely the multistage nature of the innovative process, and the multilateral-incentive problem. Firms try to overcome this agency problem by adopting stock-based grants which have become a primary component of compensation for employees in R&D departments in the past two decades. Since the researchers' actions have a great impact on the performance of the firms, which in turn affects the return of their stocks, stock-based compensation reduces the agency problem by providing a direct link between company performance and researchers' wealth, thereby providing incentive for researchers to put in effort in research. Moreover, since the stock-based grants usually have a vesting period during which they cannot be sold, they are widely used by firms to provide long-term incentives. The question then is whether these schemes are optimal.

We approach the problem by first studying the contracting problem in the abstract, deriving the optimal contract and demonstrating an implementation of the optimal contract. Finally we relate our implementation result with the observed business practices. Our finding is that the optimal contract can be implemented by using a risky security, which shares features of the stock of these firms, thereby providing a theoretical justification for the wide-spread use of stock-based compensation in firms that rely on R&D.

Briefly, setup of the article is as follows. A principal hires two risk-averse agents to perform an R&D project. At any point in time, the agents can either choose to devote effort to work or shirk; and their actions cannot be monitored by the principal, which creates a moral-hazard problem. The R&D project has multiple stages. The transition from one stage to the next is modeled by a Poisson-type process, and the arrival rate is jointly determined by the effort choice of both agents. Hence, the principal cannot treat each agent separately. To overcome the moral-hazard problem, the principal offers a long-term contract to each agent that specifies a history-contingent payment-scheme based on the information that the principal can observe. In terms of public information, we consider two scenarios. In the first scenario, which we call team-performance case, only joint performance of the team could be observed. This scenario maps into the case in which the researchers work as a team on the same innovation approach. While in the other scenario, the principal can observe each individual's performance, i.e. when an innovation is made, the principal can identify the agent who makes the discovery. This scenario captures the case in which parallel innovation are used where several new ideas are explored simultaneously. We call this scenario individual-performance case.

We use recursive techniques to characterize the optimal dynamic contract. We start with a simplified problem in which the team consists only one agent. After characterizing the optimal contract in this problem, we use similar techniques to analyze the multi-agent problem. The optimal compensation-scheme combines reward and punishment. In case of failure, the principal punishes all the agents by decreasing their payments over time. In case of success, the principal rewards all the agents in team-performance case or the agent who makes the discovery in individual-performance case by an upward jump in payment. Furthermore, in individual-performance case, since the principal can observe each agent's performance, an agent's compensation depends not only on his own performance, but may also be tied to the other agent's performance as well. We show that the optimal incentive regime depends crucially on the way in which their actions interact with one another. When their efforts are substitutes, an agent's action has a negative externality on the performance of his coworker, and hence relative-performance evaluation is used in which the principal penalizes him when his coworker succeeds. When there is complementarity between agents' efforts, the principal uses joint-performance evaluation, in which an agent also receives a reward when his coworker succeeds.

We also provide a way to implement the optimal contract for the simplified single-agent problem, in which a primary component of the agents' compensation is a state-contingent security whose return in case of success is higher than that in case of failure. We assume that investing in this security is the only saving-technology for the agents to smooth consumption over time. At any point in time, besides the effort-choice, the agent also chooses how much to consume and how much to invest in the security, subject to a minimum-holding requirement. Different from the optimal contract, in which the principal controls the agents' consumption directly, the agents choose the consumption process by themselves in this implementation, which nonetheless generates the same effort and consumption process as the optimal contract. This implementation overcomes the problem pointed out by Rogerson (1985) which is that, if the agent is allowed access to credit, he would choose to save some of his wages, if he could, because of a wedge between the agent's Euler equation and the inverse Euler equation implied by the principal's problem. In our implementation, however, the return on savings is state contingent. When we choose the state-dependent rates of return appropriately, the agent's Euler equation mimics the inverse Euler equation; put differently, the wedge between the Euler equation and the inverse Euler equation disappears.

This implementation is similar to the stock-based compensation scheme used in the real-world in two aspects. First, the return of the state contingent security and the stock price have a similar trend, with an notable increase after each breakthrough in R&D. Second, in the implementation, the agent is required to hold a certain amount of the state-contingent security until he completes the entire project. Similarly, stock-based grants usually have a vesting period during which they cannot be sold. Capturing these two main features, our implementation provides a theoretical explanation for the compensation scheme used in reality.

This article is related to four strands of literature: incentives for innovation, multi-agent incentive problem, management compensation, and dynamic contracts. There are a few articles study contracting for innovation with different focuses from ours. Manso (2011) studies a two-period model in which a principal must not only provide incentive for an agent to work rather than shirk, but also to work on exploration of an uncertain technology rather than exploitation of a known technology. Hörner and Samuelson (2012) and Bergemann and Hege (2005) study contracting problems with dynamic moral hazard and private learning about the quality of the innovation project; see also Halac, Kartik, and Liu (2012), which adds adverse selection about the agent's ability into the problem. Our study differs from these articles in three aspects. First, all these studies assume that the research project ends when there is a success, whereas in our research the project progress through distinct stages. In our setting, the multi-stage problem is not a simple repetition of single-stage problem. The optimal contract depends on the entire history of the innovation process, for example, the current stage level and the length of time it took the agents to finish each previous stage. The other common feature of these studies is that they focus on single-agent problem. By contrast, we consider a multilateral-incentive problem and study the strategic interaction between the researchers. Finally, we assume agents are risk averse instead of risk neutral. Risk aversion gives rise to a trade-off in the contracting problem. On the one hand, to introduce incentives, the principal needs to change agents' payments discontinuously after each success. On the other hand, risk-aversion suggests gains from consumption smoothing. This article describes the precise dynamic pattern of the optimal contract in which the payment is history contingent and varies over time.

This article also fits into the literature of multilateral-incentive problem. The optimal incentive regimes have been widely discussed in this literature. In a static setting, Lazear and Rosen (1981), Holmstrom (1982), and Green and Stokey (1983) give a rationale for relative-performance evaluation when the performance measures of workers have a common noise component. Che and Yoo (2001) argues that joint-performance evaluation could be used in a repeated setting because a shirking agent is punished by the subsequent shirking of his partner, which provides stronger incentive for working. However, our study shows that the type of compensation scheme that the principal should use depends crucially on how the agents' efforts interact, which sheds new light on the notion of optimal incentive regimes.

In the management-compensation literature, there is extensive research on stock-based grants for CEO compensation. For researchers' compensation, Anderson, Banker, and Ravindran (2000), Ittner, Lambert, and Larcker (2003), and Murphy (2003) have documented that executives and employees in new-economy firms receive more stock-based compensation than do their counterparts in old-economy firms. Sesil, Kroumova, Blasi, and Kruse (2002) compares the performance of 229 "New Economy" firms offering broad-based stock options to that of their non-stock option counterparts, and shows that the former have higher shareholder returns. Our implementation contributes to this literature by giving a rationale for the use of stock-based compensation in new economy firms from a theoretical point of view.

In terms of methodology, this article follows the rich and growing literature on dynamic moral hazard that uses recursive techniques to characterize optimal dynamic contracts (e.g. Green (1987), Spear and Srivastava (1987), and more recently Hopenhayn and Nicolini (1997) and Sannikov (2008)). Biais, Mariotti, Rochet, and Villeneuve (2010) and Myerson (2010) consider the dynamic moral-hazard problem in a similar continuous time and Poisson framework. Our study differs from these articles by looking at the dynamic-contracting problem in a multi-agent setup instead of single-agent environment.

The rest of the article is organized as follows. Section 2 describes the model. Section 3 analyzes a simplified case in which the research team consists only one agent. Section 4 analyzes the optimal contract for the multi-agent problem. We provide an example in which there is a closed-form solution in section 5. In section 6, we provide an implementation of the optimal dynamic contract. Section 7 concludes.

2 The Model

Time is continuous. At time 0, a principal hires two agents to perform an R&D project. The project has N stages, which must be completed sequentially. When the project is at stage n ($0 < n \leq N$), we mean that the agents have finished the $(n-1)$-th innovation and are working on the n-th innovation.

At any point in time, each agent, indexed by i ($i = 1, 2$), faces a binary-choice problem of taking an action $a_i \in A_i = \{Work, Shirk\}$. Let $A = A_1 \times A_2$ and denote a typical profile of A by $a = (a_1, a_2)$. The completion of each stage of the project is modeled by a Poisson-type process. The agents' actions jointly determine the Poisson arrival-rate in the following way. Each agent's arrival rate of making a discovery is determined by a function $R_i(a) : A \to \mathbb{R}_+$. Then, the total arrival rate of completion of each stage is $R(a) = R_1(a) + R_2(a)$. For simplicity, we assume that if agent i shirks he fails with probability 1, i.e. $R_i(Shirk) = 0$. The following table describes all the possible actions and the arrival rates for each action taken by the agents:

		Agent 2	
		Work	Shirk
Agent 1	Work	λ_1, λ_2	$\hat{\lambda}_1, 0$
	Shirk	$0, \hat{\lambda}_2$	$0, 0$

In the above table, λ_i is agent i's arrival rate when both agents exert effort, and $\hat{\lambda}_i$ is his arrival rate when he exerts effort and the other agent shirks. We assume that the probability of success increases when both agents put in effort, i.e. $\lambda = \lambda_1 + \lambda_2 > \max\{\hat{\lambda}_1, \hat{\lambda}_2\}$. To simplify notation, we use λ_{-i} and $\hat{\lambda}_{-i}$ to indicate agent i's coworker's corresponding arrival rates from now on.

Effort-choice is private information, and thus cannot be observed by the principal or the other agent. In terms of public information, we consider two different scenarios. In the first scenario, the principal can only observe the joint performance of the agents. This scenario is equivalent to the case in which the two agents work as a team. The arrival rate of success of the team is λ when both agents put in effort, $\hat{\lambda}_i$ when only agent i exerts effort or 0 when both agents shirk. We call this scenario team-performance case. Let H^t summarize all the public information up to time t. Then, H^t includes information about how many innovations were made before time t, and the exact time when each innovation was made. In the other scenario, when an innovation is made, the principal can also identify the agent who makes the discovery. In this case, besides the previous information, H^t also includes the identity of the agent who completed each innovation before time t. This scenario is called individual-performance case.

We assume that the completion of the project is sufficiently valuable to the principal that he always wants to induce both agents to work. Hence, the principal's problem is to minimize the cost of providing incentives. At time 0, the principal offers each agent a contract that specifies a flow of consumption $\{c_i^t(H^t), 0 \leq t < +\infty\}$ $(i = 1, 2)$, based on the principal's observation of their performance. Let T denote the stochastic stopping time when the last stage of the project is completed, which is endogenously determined by the agents' actions. Note that the history of H^t will not get updated after the project is completed, which implies that agents' payment-flow is constant after the completion of the project. Therefore, the principal can equivalently give the agents a lump-sum consumption transfer at T.

Each agent's utility is determined by his consumption flow and his effort. For simplicity, we assume that the two agents have the same utility function, which is further assumed to have a separable form $U(c_i) - L(a_i)$, where $U(c_i)$ is the utility from consumption and $L(a_i)$ is the disutility of exerting effort. We assume that $U : [0, +\infty) \to [0, +\infty)$ is an increasing, concave, and C^2 function with the property that $U'(c) \to +\infty$ as $c \to 0$. We also assume that the disutility of putting in effort equals some $l > 0$ and the disutility of shirking equals zero, i.e. $L(Work) = l$ and $L(Shirk) = 0$.

Given a contract, agent i's objective is to choose an effort process $\{a_i^t(H^t), 0 \leq t < \infty\}$ to maximize his total expected utility. Thus, agent i's problem is

$$\max_{\{a_i^t, 0 \leq t < \infty\}} E\left[\int_0^T re^{-rt}(U(c_i^t) - L(a_i^t))dt + e^{-rT}U(c_i^T)\right],$$

where r is the discount rate. We normalize the flow term by multiplying it by the discount rate so that the total discounted utility equals to the utility flow when the flow is constant over time. Thus, agent i's total discounted utility at time T equals $U(c_i^T)$. The agents have a reservation-utility v^0. If the maximum expected utility they can get from the contract is less than v^0, then they will reject the principal's offer.

We assume that the agents and the principal have the same discount rate. Hence, the principal's expected cost is given by

$$E\left[\int_0^T re^{-rt}(c_1^t + c_2^t)dt + e^{-rT}(c_1^T + c_2^T)\right].$$

The principal's objective is to minimize the expected cost by choosing an incentive-compatible payment scheme subject to delivering the agents the requisite initial value of expected utility v^0.

Therefore, the principal's problem is

$$\min_{\{c_1^t, c_2^t, 0 \le t < +\infty\}} E\left[\int_0^T re^{-rt}(c_1^t + c_2^t)dt + e^{-rT}(c_1^T + c_2^T)\right]$$

s.t.

$$E\left[\int_0^T re^{-rt}(U(c_i^t) - l)dt + e^{-rT}U(c_i^T)\right] \ge v^0$$

for $i = 1, 2$. We assume that the agents play a noncooperative game. Therefore, incentive compatibility in this context means that, at any point in time, each agent is willing to exert effort conditional on the other agent is putting in effort until the project is completed. In other words, exerting effort continuously is a Nash equilibrium played by these two agents[1].

Finally, to simplify the analysis, we could recast the problem as one where the principal directly transfers utility to the agents instead of consumption. In the transformed problem, the principal chooses a stream of utility transfers $\{u_i^t(H^t), 0 \le t < +\infty\}$ to minimize the expected cost of implementing positive effort. Then, the principal's problem becomes

$$\min_{\{u_1^t, u_2^t, 0 \le t < +\infty\}} E\left[\int_0^T re^{-rt}(S(u_1^t) + S(u_2^t))dt + e^{-rT}(S(u_1^T) + S(u_2^T))\right]$$

s.t.

$$E\left[\int_0^T re^{-rt}(u_i^t - l)dt + e^{-rT}u_i^T\right] \ge v^0,$$

where $S(u) = U^{-1}(u)$, which is the principal's cost of providing the agent with utility u. It can be shown that S is an increasing and strictly convex function. Moreover, $S(0) = 0$ and $S'(0) = 0$.

3 Single-agent Problem

Before analyzing the multi-agent case, we first study a simplified problem in which the R&D unit consists only one agent. Now, the agent's arrival rate of success is λ conditional on putting in effort or 0 otherwise. We demonstrate the techniques to derive the optimal dynamic contract for this simplified problem in this section and similar techniques will be used to study the more complicated multi-agent problem.

[1] In most of the cases we study, we could show that the optimal contract we derive has the property that the "{Work,Work}" equilibrium yields highest payoff for the agents among all possible equilibria. Therefore, the agents are willing to choose the equilibrium that the principal wants to implement.

To analyze the problem, we follow the standard approach in the contracting literature: the optimal contract is written in terms of the agent's continuation-utility v_t, which is the total utility that the principal expects the agent to derive at any time t. At any moment of time, given the continuation utility, the contract specifies the agent's utility flow, the continuation utility if the agent makes a discovery, and the law of motion of the continuation utility if the agent fails to make a discovery.

To derive the recursive formulation of this contracting problem, we first look at a discrete-time approximation of the continuous-time problem. The continuous-time model can be interpreted as the limit of discrete-time models in which each period lasts Δt. When Δt is small, conditional on putting in effort, the probability that the agent successfully finishes the innovation during Δt is approximately $\lambda \Delta t$. For any given continuation-utility v, the principal needs to decide a triplet $(u, \underline{v}, \bar{v})$ in each period, where

- u is the transferred utility in the current period.
- $\underline{v}$ is the next-period continuation utility if the agent fails to make a discovery during this period of time.
- $\bar{v}$ is the next-period continuation utility if the agent completes an innovation during this period of time.

If the agent chooses to exert effort, his expected utility in the current period is

$$r(u-l)\Delta t + e^{-r\Delta t}((1-\Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}),$$

where the first term is the current-period utility flow and the second term is the discounted expected continuation utility.

If the agent chooses to shirk, he does not incur any utility cost and will fail to make a discovery with probability 1. Thus, his expected utility in the current period is

$$ru\Delta t + e^{-r\Delta t}\underline{v}.$$

The triplet $(u, \underline{v}, \bar{v})$ should satisfy two conditions. First, this contract should indeed guarantee that the agent gets the promised-utility v. That is

$$r(u-l)\Delta t + e^{-r\Delta t}((1-\Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}) = v.$$

Second, the contract should implement positive effort, i.e. the expected utility of putting in effort should be higher than the expected utility of shirking. Thus,

$$r(u-l)\Delta t + e^{-r\Delta t}((1-\Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}) \geq ru\Delta t + e^{-r\Delta t}\underline{v}.$$

Let $C_n(v)$ be the principal's minimum expected cost of providing the agent with continuation-utility v when the project is at stage n. Then, $C_n(v)$ satisfies the following Bellman equation

$$C_n(v) = \min_{u,\underline{v},\bar{v}} r(S(u))\Delta t + e^{-r\Delta t}((1-\lambda\Delta t)C_n(\underline{v}) + \lambda\Delta t C_{n+1}(\bar{v}))$$

s.t.

$$\begin{aligned} r(u-l)\Delta t + e^{-r\Delta t}((1-\Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}) &= v, & (1)\\ r(u-l)\Delta t + e^{-r\Delta t}((1-\Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}) &\geq ru\Delta t + e^{-r\Delta t}\underline{v}, & (2)\end{aligned}$$

where $S(u)$ is the principal's cost given the transferred-utility u. Equation (1) is the promise-keeping condition and equation (2) is the incentive-compatibility condition.

Multiplying both sides of the Bellman equation and the promise-keeping condition (1) by $(1 + r\Delta t)/\Delta t$ and letting Δt converge to 0, we derive the following Hamilton-Jacobi-Bellman (HJB) equation in continuous time[2]

$$rC_n(v) = \min_{u,\bar{v}} rS(u) + C_n'(v)\dot{v} + \lambda(C_{n+1}(\bar{v}) - C_n(v))$$

s.t.

$$\begin{aligned} \dot{v} &= rv - r(u-l) - \lambda(\bar{v} - v), & (3)\\ \lambda(\bar{v} - v) &\geq rl. & (4)\end{aligned}$$

The promise-keeping condition (1) becomes the evolution of the agent's continuation utility (3). In the discrete-time case, after choosing u and $\bar{v}$, $\underline{v}$ is given by the promise-keeping condition. When Δt converges to 0, $\underline{v}$ converges to v. In continuous time, therefore, the continuation utility changes smoothly in case of failure, whose rate of change is determined by u and $\bar{v}$. The continuation utility

[2]In this article, we derive the HJB equation, evolution of continuation utility, and the incentive-compatibility condition in continuous time by considering the limit of a discrete-time approximation. We can also derive these formally using stochastic-calculus techniques (see Biais et al. (2010)). The reason we choose this method is because it is more intuitive and generates the same result.

can be explained as the value that the principal owes the agent. Hence, it grows at the discount-rate r and falls due to the flow of repayment $r(u-l)$ plus the gain of utility $\bar{v}-v$ at rate λ if the agent completes the innovation.

The incentive-compatibility constraint becomes a very simple expression (4). By exerting effort, the agent increases the rate of gaining of utility $\bar{v}-v$ from 0 to λ. Hence, the left-hand-side of the incentive-compatibility constraint is his benefit of exerting effort. The right-hand-side is his cost of putting in effort. In order to provide incentive, the benefit should exceed the cost. To get the agent to put in positive effort, the continuation utility should jump up by at least $\frac{rl}{\lambda}$ in case of success. The minimum reward is determined by three parameters: r, l, and λ, which have the following interpretations. (1) r is discount rate. The agent discounts the future utility at higher rate when r is bigger. (2) l measures the cost of doing research. When l is big, the cost of doing research is high. (3) λ measures the difficulty of the R&D project. Small λ implies a small chance of success. Thus, a big reward is associated with a high discount-rate, or a high cost of exerting effort, or a low chance of success.

Note that the continuation utility cannot be less than 0, because the agent can guarantee a utility level of 0 by not putting in any effort. Therefore, a negative continuation utility is not implementable.

In the HJB equation, to solve the stage-n problem, we need to know the functional form of C_{n+1}. Observe that when the last-stage innovation is completed, the cost of providing continuation-utility v (lump-sum transfer) is given by $C_{N+1}(v) = S(v)$, which is known. Hence, we solve the multi-stage problem by backward induction. First, we assume that C_{n+1} satisfies the following assumption

Assumption A: *C_{n+1} is a C^2 function. Its derivative, C'_{n+1}, is a continuous and strictly increasing function. Moreover, C'_{n+1} satisfies:*

- *$C'_{n+1}(v) \geq S'(v)$ for all $v > 0$, and $C'_{n+1}(0) = S'(0) = 0$.*

Then, we derive C_n from the HJB equation given C_{n+1}, and show that C_n also satisfies Assumption A. It is straightforward to check that C_{N+1} satisfies Assumption A. This result allows us to keep doing the backward-induction exercise until we solve the entire multi-stage problem.

To characterize the solution of the HJB equation, we do a diagrammatic analysis in the v-$C'_n(v)$ plane. The dynamics of v and $C'_n(v)$ are determined by the sign of dv/dt and $dC'_n(v)/dt$. The next two lemmas determine the sign of $dC'_n(v)/dt$ and dv/dt under two different conditions.

Lemma 3.1 *If $C_n'(v) < C_{n+1}'(v + \frac{rl}{\lambda})$, then $\frac{dC_n'(v)}{dt} < 0$ and*

$$\frac{dv}{dt}\begin{cases} < 0, & \text{if } C_n'(v) > S'(v);\\ = 0, & \text{if } C_n'(v) = S'(v);\\ > 0, & \text{if } C_n'(v) < S'(v).\end{cases}$$

Lemma 3.2 *If $C_n'(v) \geq C_{n+1}'(v + \frac{rl}{\lambda})$, then $\frac{dC_n'(v)}{dt} = 0$ and $\frac{dv}{dt} < 0$.*

The proof of these lemmas can be found in the appendix.

Lemmas 3.1 and 3.2 characterize the dynamics of v and $C_n'(v)$ in the v-$C_n'(v)$ plane. The $C_n'(v) = S'(v)$ locus determines the dynamics of v: v is decreasing over time above it and increasing over time below it. The $C_n'(v) = C_{n+1}'(v + \frac{rl}{\lambda})$ locus determines the dynamics of $C_n'(v)$: $C_n'(v)$ is constant over time above it and decreasing over time below it. The dynamics are summarized in Figure 1.

The next step is to find the optimal path in the phase diagram. From the theorem regarding the existence of a solution to a differential equation, there is an unique path from any $v_0 > 0$ to the origin (Path 1 in Figure 2). First, any path on which the state variable v diverges to infinity could be ruled out (such as Path 2). This contains the area below Path 1. In the area above Path 1, the continuation-utility v is decreasing over time. When v hits the lower bound 0, it cannot decrease any further. Thus, we must have $dv/dt \geq 0$ at $v = 0$. This condition rules out any path above Path 1 (such as Path 3) because $dv/dt < 0$ when v reaches 0 for any path in this area. Then, Path 1 is the only candidate path left in the phase diagram, and hence it is the optimal path that we are looking for. The final step is to pin down the boundary condition at $v = 0$. At this point, we have $u = 0$ and $\bar{v} = \frac{rl}{\lambda}$. Thus, when v reaches 0, the agent's continuation utility and transferred-utility flow remain at 0 until he makes a discovery. To force the agent to put in positive effort, the principal needs to increase the agent's continuation utility to $\frac{rl}{\lambda}$ when the agent completes the current stage innovation. Then, the boundary condition at $v = 0$ satisfies

$$C_n(0) = \int_{t=0}^{\infty} e^{-rt} e^{-\lambda t} \lambda C_{n+1}\left(\frac{rl}{\lambda}\right) dt = \frac{\lambda C_{n+1}\left(\frac{rl}{\lambda}\right)}{r+\lambda}.$$

To summarize, the optimal path locates between the $C_n'(v) = S'(v)$ locus and the $C_n'(v) = C_{n+1}'(v+\frac{rl}{\lambda})$ locus and reaches the lower bound of the continuation utility at the origin (Figure 2). Then, we have the following Lemma.

Lemma 3.3 *$C_n'(v) \geq S'(v)$ for all $v > 0$, and $C_n'(0) = S'(0) = 0$.*

Lemma 3.3 indicates that, if C_{n+1} satisfies Assumption A then C_n also does. This result completes the final step of the backward-induction argument.

The optimal path and the boundary condition together determine the solution of the HJB equation. The properties of the optimal dynamic contract are summarized in Proposition 3.4.

Proposition 3.4 *The optimal contract in stage n takes the following form:*

(i) The principal's expected cost at any point is given by an increasing and convex function $C_n(v)$, which satisfies

$$rC_n(v) = rS(u) + C_n'(v)(r(v-u)) + \lambda(C_{n+1}(\bar{v}) - C_n(v)),$$

and boundary condition $C_n(0) = \frac{\lambda C_{n+1}(\frac{rl}{\lambda})}{r+\lambda}$.

(ii) The transferred-utility u satisfies $S'(u) = C_n'(v)$.

(iii) When the agent completes the current stage innovation, he enters the next stage and starts with continuation-utility $\bar{v}$, which satisfies $\bar{v} = v + \frac{rl}{\lambda}$.

(iv) In case of failure to complete the innovation, the continuation-utility v decreases over time and asymptotically goes to 0.

(v) The utility-flow u has the same dynamics as continuation-utility v.

In the optimal contract, the continuation utility decreases over time in case of failure and jumps up by a fixed amount of $\frac{rl}{\lambda}$ after each success. When the agent completes the final stage, he receives a one-time transfer and his continuation becomes stationary after that. Figure 3 is a sample path of the continuation utility for a 3-stage R&D project.

Finally, for the minimum-cost functions at different stages, we have the following corollary:

Corollary 3.5 *The minimum-cost functions satisfies*

(i) $C_n(v) > C_{n+1}(v)$ for all $v \geq 0$.

(ii) $C_n'(v) > C_{n+1}'(v)$ for all $v > 0$ and $C_n'(0) = C_{n+1}'(0) = 0$.

Part (i) of Corollary 3.5 indicates that the cost of delivering the same level of continuation utility is higher when the project is at an earlier stage. The principal's problem is to minimize the cost of delivering a promised level of continuation utility while providing incentive. In order to provide incentive, the payment must change over time depending on the agent's realized performance. When the project is at an earlier stage, there are more uncertainties left in the future. Hence, the cost of delivering the same level of continuation utility to a risk-averse agent is higher at an earlier stage. Because utility flow satisfies $S'(u) = C_n'(v)$, part (ii) implies that the transferred-utility flow is also higher at an earlier stage given the same continuation utility. At any point in time, the principal has two ways to provide continuation utility: as instantaneous payment (utility flow), or as future promise. When the project is at an earlier stage, the principal chooses to provide more utility as instantaneous payment rather than future promise because, from part (i), the cost of delivering the same promised level of continuation utility is higher at an earlier stage.

4 Multi-agent Problem

We now return to the model where the R&D unit consists multiple agents and derive the optimal contracts for two different scenarios: when the principal can only observe the performance of the team and when each agent's performance could be observed.

4.1 Team Performance

First, we look at the case in which the principal can only observe joint performance of the two agents. As before, the optimal contract for agent i is written in terms of his continuation-utility v_i. At any moment of time, given v_i, the contract specifies agent i's utility-flow u_i, the continuation-utility $\bar{v}_i$ if the team makes a discovery, and the law of motion of the continuation utility if team fails.

In the multi-agent context, incentive compatibility means that agent i is willing to exert effort conditional on that the other agent is putting in effort. When his co-worker exerts effort, by putting in effort instead of shirking, agent i increases the team's arrival rate from $\hat{\lambda}_{-i}$ to λ. After success, his continuation utility jumps from v_i to $\bar{v}_i$. Thus, his benefit of exerting effort is $(\lambda - \hat{\lambda}_{-i})(\bar{v}_i - v_i)$, whereas his cost of putting in effort is rl. Hence, the contract should satisfy the following Nash-

Incentive-Compatibility (NIC) condition

$$(\lambda - \hat{\lambda}_{-i})(\bar{v}_i - v_i) \geq rl.$$

When agent i exerts effort, his continuation utility grows at the discount-rate r and falls due to the flow of repayment $r(u_i - l)$ plus the gain of utility $\bar{v}_i - v_i$ at rate λ if the team completes the innovation. Thus, his continuation utility in case of failure evolves according to

$$\dot{v}_i = rv_i - r(u_i - l) - \lambda(\bar{v}_i - v_i).$$

Let $W_n(v_1, v_2)$ be the principal's minimum cost of delivering continuation utility (v_1, v_2) when the project is at stage n. Note that agent i's NIC condition and evolution of continuation utility only depend on his own policy variables. This property implies that the cost function $W_n(v_1, v_2)$ has a separated form : $W_n(v_1, v_2) = C_{1,n}(v_1) + C_{1,n}(v_2)$, where $C_{i,n}$ is the principal's cost function of providing agent i with continuation-utility v_i when the project is at stage n. Using similar technicals in single-agent problem, we could derive the HJB equation that $C_{i,n}$ satisfies, which is

$$rC_{i,n}(v_i) = \min_{u,\bar{v}} rS(u_i) + C'_{i,n}(v_i)\dot{v}_i + \lambda(C_{i,n+1}(\bar{v}_i) - C_{i,n}(v_i))$$

s.t.

$$\begin{aligned} \dot{v}_i &= rv_i - r(u_i - l) - \lambda(\bar{v}_i - v_i), \\ (\lambda - \hat{\lambda}_{-i})(\bar{v}_i - v_i) &\geq rl. \end{aligned}$$

Note that the HJB equation for the single-agent problem is a special case of the above equation where $\hat{\lambda}_{-i} = 0$. By doing a similar diagrammatic analysis, we could characterize the solution to the HJB equation. The property of the optimal contract is summarized in the following proposition

Proposition 4.1 *At stage n $(0 < n \leq N)$, the contract for agent i that minimizes the principal's cost takes the following form:*

(i) The principal's expected cost at any point is given by an increasing and convex function $C_{i,n}(v_i)$ that satisfies the HJB equation and the boundary condition

$$C_{i,n}\left(\frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}\right) = \frac{\lambda C_{i,n+1}\left(\frac{(r+\hat{\lambda}_{-i})l}{\lambda - \hat{\lambda}_{-i}}\right)}{r + \lambda}.$$

(ii) When the team completes an innovation, agent i's continuation utility jumps to $\bar{v}_i$, which satisfies $\bar{v}_i = v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}$.

(iii) In case of failure to complete the innovation, the continuation-utility v_i is decreasing over time and asymptotically goes to $\frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$.

(iv) The utility-flow u_i has the same dynamics as continuation-utility v_i.

Different from the single-agent problem, the lower bound on implementable continuation utility in this case is a positive level: $\frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$. The positive lower-bound is due to a free-rider problem that arises when only joint performance is observable. To provide incentive, the principal should reward every agent when the team completes an innovation. Thus, even if an agent shirks, he still has a chance to get the reward when the other agent succeeds. Therefore, the principal cannot punish the agents too severely. Otherwise, an agent will choose to shirk and free ride on his coworker's success if he cannot expect to get enough payments from his contract.

The optimal contract guarantees that exerting effort continuously is a Nash equilibrium strategy played by both agents. However, in some cases, it may not be the unique equilibrium. But, it can be shown that this equilibrium yields the highest payoff for the agents and hence this "working" equilibrium is a reasonable prediction given the contract. To simplify notation, let α_i denote the effort process chosen by agent i, i.e. $\alpha_i = \{a_i^t(H^t), 0 \leq t < \infty\}$, and $E_i(\alpha_i, \alpha_{-i})$ denote the expected utility that agent i can receive under the contract when the two agents choose effort process (α_i, α_{-i}). Let $(\alpha_i^w, \alpha_{-i}^w)$ be the choice that both agents exert effort until the completion of the project. Now, consider another effort choice $(\tilde{\alpha}_i, \tilde{\alpha}_{-i})$ in which the agents choose to shirk under some circumstances. If at some point in time agent i's coworker chooses to shirk, this action eliminate agent i's chance of free-riding. Because it decreases agent i's probability of jumping to a higher-payoff path, we have $E_i(\tilde{\alpha}_i, \tilde{\alpha}_{-i}) < E_i(\tilde{\alpha}_i, \alpha_{-i}^w)$. Furthermore, the NIC condition implies that agent i is always willing to exert effort when his coworker puts in effort. It follows that $E_i(\tilde{\alpha}_i, \alpha_{-i}^w) \leq E_i(\alpha_i^w, \alpha_{-i}^w)$. Therefore, $E_i(\tilde{\alpha}_i, \tilde{\alpha}_{-i}) < E_i(\alpha_i^w, \alpha_{-i}^w)$, which means the effort precess $(\tilde{\alpha}_i, \tilde{\alpha}_{-i})$ gives lower expected utility for agent i than $(\alpha_i^w, \alpha_{-i}^w)$ does. This result shows that even if there exist some other equilibria, they yield less payoff for both agents. Due to this reason, the agents are willing to choose the equilibrium that the principal wants to implement.

4.2 Individual Performance

Next, we derive the optimal contract for the case in which the principal can observe each agent's performance. Now, because the principal can identify the agent who completes the innovation when there is a breakthrough, agent i's compensation not only depends on his own performance but also relates to his coworker's performance. Given continuation utility v_i, agent i's contract specifies his utility-flow u_i, his continuation-utility $\bar{v}_{i,i}$ if he makes a discovery, his continuation-utility $\bar{v}_{i,-i}$ if his coworker makes a discovery, and the law of motion of his continuation utility if both agents fail.

By putting in effort, agent i increase his own arrival rate of success from 0 to λ_i and changes his coworker's arrival rate from $\hat{\lambda}_{-i}$ to λ_{-i}. Therefore, $\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i)$ is his benefit of putting in effort. His cost of exerting effort is still rl. Hence, the NIC condition in this case is

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) \geq rl.$$

The sign of $(\lambda_{-i} - \hat{\lambda}_{-i})$ in the NIC condition has very important implications on the optimal contract. Recall that λ_{-i} is the arrival rate of the event that agent i's coworker makes a discovery when both agents put in effort, and $\hat{\lambda}_{-i}$ is the arrival rate of the event that agent i's coworker makes a discovery when agent i shirks while his coworker exerts effort. When $\lambda_{-i} = \hat{\lambda}_{-i}$, the efforts of agent i and the efforts of his coworker are independent because agent i's action does not affect his coworker's performance. When $\lambda_{-i} < \hat{\lambda}_{-i}$, their efforts are substitutes. When agent i chooses to exert effort instead of shirking, this action decreases his coworker's arrival rate from $\hat{\lambda}_{-i}$ to λ_{-i}. In other words, his action has a negative externality on his coworker's performance. Finally, when $\lambda_{-i} > \hat{\lambda}_{-i}$, their efforts are complements. If agent i works hard, he also increases his coworker's arrival rate from $\hat{\lambda}_{-i}$ to λ_{-i}. Thus, agent i's efforts have positive externalities on his coworker's performance.

In case of failure, the continuation utility grows at the discount-rate r. It falls due to the flow of repayment $r(u_i - l)$, the gain of utility $\bar{v}_{i,i} - v_i$ at rate λ_i if agent i completes the innovation, and the gain of utility $\bar{v}_{i,-i} - v_i$ at rate λ_{-i} if his coworker completes the innovation. Hence, agent i's continuation utility evolves according to

$$\dot{v}_i = rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i)$$

Let $W_n(v_1, v_2)$ be the principal's minimum cost of delivering continuation utility (v_1, v_2) when the project is at stage n. Similar to the team-performance case, $W_n(v_1, v_2)$ has a separated form:

$W_n(v_1, v_2) = C_{1,n}(v_1) + C_{1,n}(v_2)$, where $C_{i,n}$ is determined by the following HJB equation

$$rC_{i,n}(v_i) = \min_{u_i, \bar{v}_{i,i}, \bar{v}_{i,-i}} rS(u_i) + C'_{i,n}(v_i)\dot{v}_i - \lambda C_{i,n}(v_i) + \lambda_i C_{i,n+1}(\bar{v}_{i,i}) + \lambda_{-i} C_{i,n+1}(\bar{v}_{i,-i})$$

s.t.

$$\dot{v}_i = rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i),$$
$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) \geq rl \text{ (NIC)}.$$

Unlike the team-performance case, the principal now needs to choose two continuation-utilities $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ simultaneously to provide incentive optimally, which makes the problem much more complicated than before. However, we still could use similar diagrammatic analysis to characterize the solution, which can be found in the appendix. The main properties of optimal contract are given by the following proposition:

Proposition 4.2 *The contract that minimizes the principal's cost takes the following form:*

(i) The principal's expected cost of delivering continuation utility v_i at stage n is given by a convex function $C_{i,n}(v_i)$ that solves the HJB equation and satisfies the boundary condition

$$C_{i,n}(0) = \frac{\lambda_i C_{i,n+1}(\frac{rl}{\lambda_i}) + \lambda_{-i} C_{i,n+1}(0)}{r + \lambda}.$$

(ii) If agent i completes the innovation, then his utility-flow jumps up.

(iii) If agent i's coworker completes the innovation, then: 1) his utility-flow does not change if $\lambda_{-i} = \hat{\lambda}_{-i}$; 2) his utility-flow drops down if $\lambda_{-i} < \hat{\lambda}_{-i}$; 3) his utility-flow jumps up if $\lambda_{-i} > \hat{\lambda}_{-i}$.

(iv) If both agents fail to complete the project, agent i's continuation-utility v_i and utility-flow u_i are decreasing over time and v_i asymptotically goes to 0.

In the optimal contract, the principal rewards agent i when he makes an innovation by an upward jump in his payment. In our setup, agent i has a chance to make a discovery only when he puts in effort. Thus, a discovery by him indicates that he is exerting effort, and therefore he should be rewarded.

Regarding how an agent's payment depends on his coworker's performance, Part (iii) of Proposition 4.2 demonstrates the way in which the optimal incentive regime is a function of how agents' efforts interact with one another, which is one of the main results of this article. When $\lambda_{-i} < \hat{\lambda}_{-i}$, the principal uses relative-performance evaluation in which he punishes agent i by decreasing his payment-flow when his coworker makes a discovery. The intuition for using relative-performance evaluation is the following. In this case, agent i's efforts have negative externalities on his coworker's performance. Thus, when his coworker makes a discovery, this event provides suggestive information that agent i is shirking. Therefore, the principal should punish agent i for not putting in effort. On the contrary, when $\lambda_{-i} > \hat{\lambda}_{-i}$, agent i's efforts have positive externalities on his coworker's performance. The event that his coworker achieves a success gives an indication that agent i is also exerting effort. Therefore, the principal uses joint-performance evaluation in which he rewards agent i by an upward jump in his payment-flow when his coworker makes a discovery. Finally, when $\lambda_{-i} = \hat{\lambda}_{-i}$, because agent i's action does not affect his coworker's performance, the event that his coworker makes a discovery does not offer any useful information about whether agent i puts in effort or not. Hence, agent i's payment-flow remains the same when his coworker makes a discovery.

Finally, when joint-performance evaluation or independent-performance evaluation is used, an agent can benefit from or is not affected by his coworker's success. Thus, even if some other equilibrium exist, they cannot yield higher payoff for the agents than the "working" equilibrium due to the same reason discussed at the end of team-performance case. Therefore, we still have unique prediction of the equilibrium that the agents are going to choose. However, this result does not hold when relative-performance evaluation is used by the principal. In this case, an agent is punished when his college performs well. If both agents choose to shirk, they can avoid this punishment and get higher expected utility. Hence, the principal needs to be careful when using the relative-performance evaluation because it may induce "shirking" equilibrium.

5 Example with Closed-form Solution

In this section, we provide an example for which we obtain a closed-form solution. In fact, in this example, we can handle the case in which the project has infinitely many stages. We consider the most complicated case: the individual-performance case, and the other two cases are its special cases. When the project has infinite stages, the minimum-cost function no longer depends the stage level. The optimal contract for agent i is characterized by the following HJB equation

$$rC_i(v_i) = \min_{u_i,\bar{v}_{i,i},\bar{v}_{i,-i}} rS(u_i) + C_i'(v_i)\dot{v}_i - \lambda C_i(v_i) + \lambda_i C_i(\bar{v}_{i,i}) + \lambda_{-i} C_i(\bar{v}_{i,-i})$$

s.t.

$$\dot{v}_i = rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i),$$

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) \geq rl.$$

We assume that the utility function is logarithmic $U(c_i) = \log c_i$ and solve this HJB equation by guess-and-verify. First, note that, for logarithmic-utility function, the cost of providing transferred utility-flow u_i is given by $S(u_i) = e^{u_i}$. Inspired by this functional form, we make a guess that the cost function takes the form of qe^{v_i} $(q > 0)$—a constant times e^{v_i}. Then, using this guess, we solve the minimization problem on the right-hand side of the HJB equation. If the right-hand side also takes the form of a constant times e^{v_i}, then this guess is verified and we can pin down the constant q from the HJB equation.

Taking $C_i(v_i) = qe^{v_i}$ into the right-hand side of the HJB equation, we have

$$RHS = \min_{u_i, \bar{v}_{i,i}, \bar{v}_{i,-i}} re^{u_i} + qe^{v_i}\dot{v}_i - \lambda qe^{v_i} + \lambda_i qe^{\bar{v}_{i,i}} + \lambda_{-i} qe^{\bar{v}_{i,-i}}$$

s.t.

$$\dot{v}_i = rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i),$$

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) \geq rl.$$

Utility-flow u_i satisfies the first-order condition $S'(u_i) = C_i'(v_i)$. Therefore,

$$e^{u_i} = qe^{v_i},$$

which implies $u_i = v_i + \log q$.

The NIC condition must be blinding, otherwise first-order conditions imply that $\bar{v}_{i,i} = \bar{v}_{i,-i} = v_i$, which violates the NIC condition. Thus, $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are determined by the following system

$$\lambda_i qe^{\bar{v}_{i,i}} - \lambda_i qe^{v_i} + \gamma\lambda_i = 0 \tag{5}$$

$$\lambda_{-i} qe^{\bar{v}_{i,-i}} - \lambda_{-i} qe^{v_i} + \gamma(\lambda_{-i} - \hat{\lambda}_{-i}) = 0 \tag{6}$$

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) - rl = 0 \tag{7}$$

where (5) and (6) are first-order conditions, and (7) is the NIC condition. γ, the Lagrangian multiplier, satisfies $\gamma < 0$.

If $\lambda_{-i} = \hat{\lambda}_{-i}$, then it follows from (6) and (7) that

$$\bar{v}_{i,i} = v_i + \frac{rl}{\lambda_i},$$

$$\bar{v}_{i,-i} = v_i.$$

If $\lambda_{-i} \neq \hat{\lambda}_{-i}$, define $\Delta v_{i,i} = \bar{v}_{i,i} - v_i$ and $\Delta v_{i,-i} = \bar{v}_{i,-i} - v_i$. Combining (5) and (6), we could get

$$\frac{e^{\Delta v_{i,i}} - 1}{e^{\Delta v_{i,-i}} - 1} = \frac{\lambda_{-i}}{\lambda_{-i} - \hat{\lambda}_{-i}}. \tag{8}$$

Equation (7) could be rewritten as

$$\lambda_i \Delta v_{i,i} + (\lambda_{-i} - \hat{\lambda}_{-i})\Delta v_{i,-i} - rl = 0. \tag{9}$$

Then, $\Delta v_{i,i}$ and $\Delta v_{i,-i}$ are uniquely pined down by (8) and (9). Note that neither (8) nor (9) contains v_i, which implies that both $\Delta v_{i,i}$ and $\Delta v_{i,-i}$ depend only on the parameters of the model and are independent of the state-variable v_i. Consequently, for all cases, we have $\bar{v}_{i,i} = v_i + \Delta v_{i,i}$ and $\bar{v}_{i,i} = v_i + \Delta v_{i,-i}$, where both $\Delta v_{i,i}$ and $\Delta v_{i,-i}$ are independent of v_i.

Taking the solution for u_i, $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ into the right-hand side of the HJB equation, it becomes

$$\begin{aligned} RHS &= re^{v_i + \log q} + qe^{v_i}(-r\log q - \hat{\lambda}_{-i}\Delta v_{i,-i}) - \lambda q e^{v_i} + \lambda_i q e^{v_i + \Delta v_{i,i}} + \lambda_{-i} q e^{v_i + \Delta v_{i,-i}} \\ &= (rq + q(-r\log q - \hat{\lambda}_{-i}\Delta v_{i,-i}) - \lambda q + \lambda_i q e^{\Delta v_{i,i}} + \lambda_{-i} q e^{\Delta v_{i,-i}})e^{v_i} \end{aligned}$$

This result verifies that the right-hand side also takes the form of a constant times e^{v_i}. Finally, letting the left-hand side of the HJB equation equal to the right-hand side, we have

$$rq = rq + q(-r\log q - \hat{\lambda}_{-i}\Delta v_{i,-i}) - \lambda q + \lambda_i q e^{\Delta v_{i,i}} + \lambda_{-i} q e^{\Delta v_{i,-i}}.$$

Solving q, we get

$$q = \exp\left(\frac{\lambda_i e^{\Delta v_{i,i}} + \lambda_{-i} e^{\Delta v_{i,-i}} - \lambda - \hat{\lambda}_{-i}\Delta v_{i,-i}}{r}\right).$$

The above computation provides the solution to the HJB equation[3]. Next, we derive some

[3]For team-performance case, we have $\Delta v_{i,i} = \Delta v_{i,-i} = \frac{rl}{\lambda - \hat{\lambda}_{-i}}$ and hence

$$q = \exp\left(\frac{\lambda e^{\frac{rl}{\lambda - \hat{\lambda}_{-i}}} - \lambda - \frac{rl\hat{\lambda}_{-i}}{\lambda - \hat{\lambda}_{-i}}}{r}\right).$$

For single-agent problem, we have $\hat{\lambda}_{-i} = 0$ and hence

$$q = \exp\left(\frac{\lambda e^{\frac{rl}{\lambda}} - \lambda}{r}\right).$$

properties of the optimal contract implied by this solution. First, it follows from from (5) that $\bar{v}_{i,i} > v_i$, which means that the principal rewards agent i by an upward jump in continuation utility when he makes a discovery. From (6), we have

$$\bar{v}_{i,-i} \begin{cases} < v_i, & \text{if } \lambda_{-i} < \hat{\lambda}_{-i}; \\ = v_i, & \text{if } \lambda_{-i} = \hat{\lambda}_{-i}; \\ > v_i, & \text{if } \lambda_{-i} > \hat{\lambda}_{-i}. \end{cases}$$

Thus, when $\lambda_{-i} = \hat{\lambda}_{-i}$, agent i's continuation utility does not depends on the other agent's performance; when $\lambda_{-i} < \hat{\lambda}_{-i}$, agent i is punished by a downward jump in continuation utility when his coworker succeeds (relative-performance evaluation); and when $\lambda_{-i} > \hat{\lambda}_{-i}$ agent i is rewarded by an upward jump in continuation utility when his coworker succeeds (joint-performance evaluation). Finally, the evolution of continuation utility in case of failure follows

$$\begin{aligned} \dot{v}_i &= rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i) \\ &= r(v_i - u_i) - \hat{\lambda}_{-i}(\bar{v}_{i,-i} - v_i) \\ &= \lambda - \lambda_i e^{\Delta v_{i,i}} - \lambda_{-i} e^{\Delta v_{i,-i}} \\ &= \lambda_i(1 - e^{\Delta v_{i,i}}) + \lambda_{-i}(1 - e^{\Delta v_{i,-i}}) \\ &= (\lambda - \hat{\lambda}_{-i})(1 - e^{\Delta v_{i,i}}) \\ &< 0, \end{aligned}$$

where the third equality is obtained by substituting u_i with $v_i + \log q$ and using the expression for q and the fifth equality is obtained by using (8). This result implies that agent i's continuation utility decreases over time when both agents fail. Finally, $u_i = v_i + \log q$ implies that the utility-flow u_i has the same dynamics as the continuation-utility v_i. The properties of the optimal contract are summarized in the following proposition.

Proposition 5.1 *The optimal contract has the following properties:*

(i) If agent i completes the innovation, then his utility-flow jumps up.

(ii) If agent i's coworker completes the innovation, then: 1) agent i's utility-flow does not change if $\lambda_{-i} = \hat{\lambda}_{-i}$; 2) his utility-flow drops down if $\lambda_{-i} < \hat{\lambda}_{-i}$; 3) his utility-flow jumps up if $\lambda_{-i} > \hat{\lambda}_{-i}$.

(iii) If both agents fail to complete the project, agent i's continuation-utility v_i and utility-flow u_i decrease over time.

Notice that all the properties are consistent with those of the optimal contract discussed in Section 4.2, and the intuition behind these results are also the same.

6 Implementation

The optimal contract in the previous sections is written in terms of continuation utility, which is highly abstract. Moreover, the principal controls the agents' consumption directly, i.e. the agents consume all the payments from the principal at any point in time. In this section, we provide an implementation of the optimal contract, in which a primary component of the agents' compensation is a state-contingent security. In this implementation, besides the decision of exerting effort or shirking, the agents also make consumption decision by themselves. Yet, the implementation generates the same consumption allocation as the original optimal contract. Finally, we briefly discuss how this implementation relates to the compensation schemes used in reality.

For simplicity, we demonstrate the implementation for the single-agent problem and then discuss how it could be extended to the multi-agent problem. To introduce the design of the state-contingent security, we first look at a discrete-time approximation of the continuous-time setting. In each period, the agent has access to a security which lasts for one period. When the project is at stage n, y shares of this security bought in period t pays y in period $t+1$ if the agent fails to make a discovery. If the agent succeeds, the payoff is $Y_{n+1}(y)$, where $Y_{n+1}(y)$ is a function of y. The price of the security is determined by fair-price rule, i.e. the price of the security equals the present value of this security. Let $P_n(y)$ denote the price of y shares of the security when the project is at stage n. Then[4],

$$P_n(y) = e^{-r\Delta t}((1-\lambda\Delta t)y + \lambda\Delta t Y_{n+1}(y)).$$

To implement the optimal contract, before the project starts, the principal provides the agent with initial-wealth y^0, and $\underline{y}_1$ ($\underline{y}_1 \leq y^0$) of the initial wealth is paid in terms of this security. When the project proceeds, the agent is required to hold a minimum amount of this security, denoted by

[4] In general, the pricing function P_n is non-linear. But, if the utility function is logarithmic, then $Y_{n+1}(y)$ is a linear function of y, and hence the pricing function becomes linear $P_n(y) = p_n y$, where p_n is the price for each share of the security.

$\underline{y}_n$ depending on the stage level, until the whole project is completed. We assume that investing in this security is the only saving technology for the agent to carry wealth over time. Hence, besides effort choice, the agent also decides how much to consume and how much to invest in the security. Let y_t denote the agent's wealth in period t. Then, his budget constraint is

$$rc_t\Delta t + P_n(y_{t+1}) \leq y_t,$$

where the first term on the left-hand side is his consumption in the current period, and the second term is his investment in the security. Note that y_{t+1} is the number of shares of the security that the agent purchases in period t, which is also his wealth in period $t+1$ if he fails to make a discovery. Let Δt converges to 0, we can derive the evolution of the agent's wealth in case of failure, which satisfies

$$\dot{y} = ry - rc - \lambda(Y_{n+1}(y) - y).$$

When the project is at stage n, the agent's wealth in case of failure grows at rate r, and decreases due to the spending on consumption c, and the loss of the investment in the security $\lambda(Y_{n+1}(y)-y)$. If the agent succeeds, his wealth jumps to $Y_{n+1}(y)$.

Now, the agent's problem is to choose an effort process and a consumption process to maximize his discounted expected utility. Let $V_n(y)$ be the maximum expected utility that the agent can get given income y when the project is at stage n. In recursive form, the agent's problem is to solve the following HJB equation

$$rV_n(y) =$$
$$\max_a\{\max_c r(U(c)-l) + V_n'(y)\dot{y} + \lambda(V_{n+1}(Y_{n+1}(y)) - V_n(y)), \max_c rU(c) + V_n'(y)\dot{y}\}$$

s.t.

$$\begin{aligned} \dot{y} &= ry - rc - \lambda(Y_{n+1}(y) - y), \\ y &\geq \underline{y}_n. \end{aligned}$$

The next proposition shows that this implementation generates the same consumption allocation and effort choice as the original optimal contract when the principal sets initial wealth, payoff in case of success, and minimum holding requirement appropriately. The proof is in the appendix.

Proposition 6.1 *Suppose the principal provides the agent with initial-wealth y^0*

$$y^0 = C_1(v^0),$$

and at stage n

$$Y_{n+1}(y) = C_{n+1}\left(C_n^{-1}(y) + \frac{rl}{\lambda}\right),$$

$$\underline{y}_n = C_n(0).$$

Then, given income y, the highest discounted expected utility the agent can get is

$$V_n(y) = C_n^{-1}(y).$$

Moreover, he chooses the consumption process as the one in the optimal contract and always exerts effort until he completes the last-stage innovation.

The idea of this implementation comes from the fact that the agent's utility maximization problem is the dual problem of the principal's cost minimization problem in section 3. Given continuation-utility v, $C_n(v)$ is the minimum expected cost to finance the incentive-compatible compensation scheme. From the dual perspective, given expected wealth $y = C_n(v)$, the maximum expected utility that the agent can reach under an incentive-compatible compensation scheme should equal v. Furthermore, the consumption allocation should be the same.

For multi-agent problem, to implement the optimal contract for team-performance, the principal should set

$$Y_{i,n+1}(y) = C_{i,n+1}\left(C_{i,n}^{-1}(y) + \frac{rl}{\lambda - \hat{\lambda}_i}\right),$$

and

$$\underline{y}_{i,n} = C_{i,n}\left(\frac{\hat{\lambda}_{-i}l}{\lambda - \hat{\lambda}_{-i}}\right).$$

For individual-performance case, this idea requires a security whose payment depends on each individual's performance, which we cannot find a counterpart in the financial market. However, the principal could still use the implementation for team-performance case to provide incentive. But it is not optimal, because it does not capture the interaction of the agents' actions.

In this implementation, the state-contingent security plays a key role in providing incentives. The gap between the payoff in case of success and that in case of failure guarantees that the agent is willing to exert effort. The minimum holding requirement is the lowest amount of the security that can provide incentive for exerting effort. In the financial market, there does not exist such an exotic asset that has the exact same payoff structure as the state-contingent security used in this implementation. However, the stock of a company is a reasonable proxy for this security. Since these firms rely intensely on R&D, the performance of the employees in the R&D units have a great impact on these firms' performance outcomes, which bring a close relationship between employees' performance and the return of firms' stocks. In particular, after each breakthrough in R&D, it always follows a notable increase in the firm's stock price. When there is no arrival of such good news for a period time, its stock price tends to decline. Thus, among all available assets, the company's stock has the closest payoff-pattern to that of the state-contingent security. Another feature of our implementation is the minimum amount holding requirement that the agents have to meet until the completion of the project. In the real-world, this feature is mimicked by using employee restricted-stocks, which have a vesting period during which they cannot be sold. The time restriction provides long-term incentives to overcome the repeated moral-hazard problem.

In the past two decades, stock-based grants have become the most popular compensation scheme used by new-economy firms. The similarities between this compensation scheme and our implementation of the optimal contract suggest that firms are getting as close to optimality as is allowed by the market structure. In other worlds, our implementation gives a justification for the wide-spread use of stock-based compensation in firms that rely on R&D from a theoretical point of view.

7 Conclusion

This article studies the agency problem between a firm and its in-house R&D unit. The problem is analyzed in a set-up that captures two distinct aspects, namely the multilateral feature and the multi-stage feature, in this specific agency relationship. We use recursive techniques to characterize the dynamic contract that overcome the multi-agent repeated moral-hazard problem. In the optimal contract, the principal provides incentive combining punishment and reward. He decreases every agent's payment if no discovery is made. In case of success, he rewards all the agents if he can only observe the performance of the team or the agent who makes the discovery if he can identify the agent by an upward jump in payment. Moreover, in individual-performance case, agents' payments not only depend on their own performances, but also may be tied to their peers' performances as well. Relative-performance evaluation is used if their efforts are substitutes whereas joint-performance evaluation is used if their efforts are complements. This feature of the optimal compensation scheme in our set-up provides a new viewpoint on optimal incentive regimes used in multi-agent contracting problems.

We also show that the optimal contract could be implemented by a risky security, whose return depends on the outcome of the project. The agents are required to hold a minimum amount of this security until the project is completed. In this implementation, instead of the principal directly controlling the agents' consumption as in the optimal contract, the agents choose consumption level by themselves. We show that this implementation yields the same consumption allocation and effort choice as the optimal contract. This implementation provides a theoretical justification for the stock-based compensation used in reality.

Appendix

A Proofs for Single-agent Problem

Proof of lemma 3.1:

First, we show that the incentive-compatibility constraint binds in this case. Otherwise, $\bar{v}$ is chosen to minimize $-C'_n(v)\bar{v} + C_{n+1}(\bar{v})$ and satisfies the following first-order condition $C'_n(v) = C'_{n+1}(\bar{v})$. But then $C'_n(v) < C'_{n+1}(v + \frac{rl}{\lambda})$ implies that $\bar{v} < v + \frac{rl}{\lambda}$, which violates the incentive-compatibility condition.

When the incentive-compatibility condition is binding, the rate of change of v satisfies $\frac{dv}{dt} = r(v - u)$. Therefore, the HJB equation becomes

$$rC_n(v) = \min_u rS(u) + C'_n(v)(r(v-u)) + \lambda(C_{n+1}(v + \frac{rl}{\lambda}) - C_n(v)).$$

By the envelope theorem, we can derive that

$$rC'_n(v) = rC'_n(v) + C''_n(v)\frac{dv}{dt} + \lambda C'_{n+1}(v + \frac{rl}{\lambda}) - \lambda C'_n(v).$$

Thus,

$$\frac{dC'_n(v)}{dt} = \lambda(C'_n(v) - C'_{n+1}(v + \frac{rl}{\lambda})).$$

Because $C'_n(v) < C'_{n+1}(v + \frac{rl}{\lambda})$, it follows that $\frac{dC'_n(v)}{dt} < 0$.

As $\frac{dv}{dt} = r(v-u)$, the sign of $\frac{dv}{dt}$ is determined by the values of v and u. Note that u is chosen to minimize $S(u) - C'_n(v)u$, which is a strictly convex function of u. Hence, u is given by the first-order condition $C'_n(v) = S'(u)$. If $C'_n(v) = S'(v)$, then $S'(v) = S'(u)$. Because $S(v)$ is strictly convex, we have $v = u$ and $\frac{dv}{dt} = r(v - u) = 0$. Similarly, $C_n'(v) > S'(v)$ implies that $\frac{dv}{dt} < 0$ and $C_n'(v) < S'(v)$ implies that $\frac{dv}{dt} > 0$ *Q.E.D.*

Proof of lemma 3.2:

In this case, it can be shown that $\bar{v}$ is determined by the first-order condition $C_n'(v) = C_{n+1}'(\bar{v})$ and $C_n'(v) \geq C_{n+1}'(v + \frac{rl}{\lambda})$ implies that $\bar{v} \geq v + \frac{rl}{\lambda}$. The incentive-compatibility condition is automatically satisfied. Then, the HJB equation becomes

$$rC_n(v) = \min_{u,\bar{v}} rS(u) + C_n'(v)(rv - r(u - l) - \lambda(\bar{v} - v)) + \lambda(C_{n+1}(\bar{v}) - C_n(v)).$$

From the envelope theorem

$$rC_n'(v) = (r + \lambda)C_n'(v) + C_n''(v)\frac{dv}{dt} - \lambda C_n'(v).$$

Therefore,

$$\frac{dC_n'(v)}{dt} = 0.$$

For the dynamics of v, note that in this case

$$\begin{aligned} \frac{dv}{dt} &= rv - r(u - l) - \lambda(\bar{v} - v) \\ &= r(v - u) + (rl + \lambda v - \lambda\bar{v}). \end{aligned}$$

Because $\bar{v} \geq v + \frac{rl}{\lambda}$, the second term is non-positive. For the first term, u is determined by the first order condition $C_n'(v) = S'(u)$. Because $C_n'(v) \geq C_{n+1}'(v + \frac{rl}{\lambda})$, we have $S'(u) \geq C_{n+1}'(v + \frac{rl}{\lambda}) \geq S'(v + \frac{rl}{\lambda})$, which implies that $u \geq v + \frac{rl}{\lambda} > v$. Thus, the first term is strictly negative. It follows that $\frac{dv}{dt} < 0$. *Q.E.D.*

Proof of proposition 3.4:

For part (i), it has been shown that $C_n(v)$ is determined by the HJB equation and the boundary condition. On the optimal path, $C_n'(v)$ is strictly increasing in v, which implies that $C_n(v)$ is strictly convex. In addition, $C_n'(0) = S'(0) = 0$. Thus $C_n'(v) > 0$ for all $v > 0$. Consequently, $C_n(v)$ is an increasing function.

Part (ii) is due to the fact that the transferred-utility flow is determined by the first-order condition $S'(u) = C_n'(v)$.

For part (iii), note that the optimal path locates in the area where the incentive-compatibility constraint binds. Hence, $\bar{v} = v + \frac{rl}{\lambda}$.

For part (iv), note that on the optimal path v is decreasing over time and asymptotically converges to 0.

Finally, from part (ii), $S'(u) = C_n'(v)$. Because $S(u)$ and $C_n(v)$ are both convex, u and v are positively related. From part (iii), $\bar{v} = v + \frac{rl}{\lambda}$. Thus, u and $\bar{v}$ have the same dynamics as v, which proves part (v). *Q.E.D.*

Proof of corollary 3.5:

Suppose the statement of Corollary 3.5 is true for some stage n. Then we have: (i) $C_n(v) > C_{n+1}(v)$ for all $v \geq 0$; (ii) $C_n'(v) > C_{n+1}'(v)$ for all $v > 0$ and $C_n'(0) = C_{n+1}'(0) = 0$. On the optimal path, $dC_n'(v)/dt = \lambda(C_n'(v) - C_{n+1}'(v + \frac{rl}{\lambda}))$ and $dv/dt = r(v - u_n)$, where u_n satisfies $S'(u_n) = C_n'(v)$. Hence, the slope of C_n' at v satisfies

$$\frac{dC_n'(v)}{dv} = \frac{\lambda(C_n'(v) - C_{n+1}'(v + \frac{rl}{\lambda}))}{r(v - u_n)}$$

Similarly, for C_{n-1}', we have

$$\frac{dC_{n-1}'(v)}{dv} = \frac{\lambda(C_{n-1}'(v) - C_n'(v + \frac{rl}{\lambda}))}{r(v - u_{n-1})}$$

where u_{n-1} satisfies $S'(u_{n-1}) = C_{n-1}'(v)$.

Suppose $C_{n-1}'(v) = C_n'(v)$ at some v. Then at this continuation-utility level, $S'(u_n) = S'(u_{n-1})$ and hence $u_n = u_{n-1}$. Furthermore, because $C_n'(v + \frac{rl}{\lambda}) > C_{n+1}'(v + \frac{rl}{\lambda})$, it follows that

$$\frac{dC_{n-1}'(v)}{dv} > \frac{dC_n'(v)}{dv}.$$

Thus, if C_{n-1}' and C_n' intersect, then C_{n-1}' is above C_n' in small interval right to the intersection and below C_n' in small interval left to the intersection. This result implies that C_{n-1}' intersects C_n' at most once. Moreover, we have shown that $C_{n-1}'(0) = C_n'(0) = 0$. Therefore, $C_{n-1}'(v) > C_n'(v)$ for all $v > 0$. A direct implication of this result is that $C_{n-1}(v) > C_n(v)$ for all v because $C_{n-1}(0) = \frac{\lambda C_n(\frac{rl}{\lambda})}{r+\lambda} > \frac{\lambda C_{n+1}(\frac{rl}{\lambda})}{r+\lambda} = C_n(0)$. Hence, the statement of Corollary 3.5 is also true for stage $n-1$.

At the final stage N, we have $C_N'(v) > C_{N+1}'(v) = S'(v)$ for all $v > 0$. It also implies that $C_N(v) > C_{N+1}(v)$ for all $v \geq 0$ because $C_N(0) = \frac{\lambda S(\frac{rl}{\lambda})}{r+\lambda} > 0 = S(0) = C_{N+1}(0)$. These results verify that the statement of Corollary 3.5 is true for $n = N$. Then, by backward induction, we can show that the statement of Corollary 3.5 is true for every stage n $(0 < n \leq N)$.

Q.E.D.

B Proofs for Multi-agent Problem

The HJB equation of individual-performance case is

$$rC_{i,n}(v_i) = \min_{u_i,\bar{v}_{i,i},\bar{v}_{i,-i}} rS(u_i) + C'_{i,n}(v_i)\dot{v}_i - \lambda C_{i,n}(v_i) + \lambda_i C_{i,n+1}(\bar{v}_{i,i}) + \lambda_{-i} C_{i,n+1}(\bar{v}_{i,-i})$$

s.t.

$$\dot{v}_i = rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i),$$

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) \geq rl \text{ (NIC)}.$$

Similar to the single-agent case, we solve this problem by backward induction. First, we assume that $C_{i,n+1}$ satisfies the following assumption

Assumption B: *$C_{i,n+1}$ is a C^2 function. Its derivative, $C'_{i,n+1}$, is a continuous and strictly increasing function. Moreover, $C'_{i,n+1}$ satisfies:*

(i) If $\lambda_{-i} \leq \hat{\lambda}_{-i}$, then $C'_{i,n+1}(v_i) \geq S'(v_i)$ for all $v_i > 0$, and $C'_{i,n+1}(0) = S'(0) = 0$.

(ii) If $\lambda_{-i} > \hat{\lambda}_{-i}$, then $C'_{i,n+1}(v_i) > S'(v_i - \frac{\hat{\lambda}_{-i}l}{\lambda - \hat{\lambda}_{-i}})$ for all $v_i \geq \frac{\hat{\lambda}_{-i}l}{\lambda - \hat{\lambda}_{-i}}$.

Then, we derive $C_{i,n}$ from the HJB equation given $C_{i,n+1}$, and show that $C_{i,n}$ also satisfies Assumption B. It is straightforward to check that $C_{i,N+1} = S$ satisfies Assumption B. This result allows us to keep doing the backward-induction exercise until we solve the entire multi-stage problem.

To characterize the solution of the HJB equation, we do a diagrammatic analysis in the v_i-$C'_{i,n}(v_i)$ plane. Given a point $(v_i, C'_{i,n}(v_i))$ in this plane, $(u_i, \bar{v}_{i,i}, \bar{v}_{i,-i})$ are determined by the following Kuhn-Tucker conditions:

$$S'(u_i) - C'_{i,n}(v_i) + \eta_1 = 0, \tag{10}$$

$$\lambda_i C'_{i,n+1}(\bar{v}_{i,i}) - \lambda_i C'_{i,n}(v_i) + \gamma\lambda_i + \eta_2 = 0, \tag{11}$$

$$\lambda_{-i} C'_{i,n+1}(\bar{v}_{i,-i}) - \lambda_{-i} C'_{i,n}(v_i) + \gamma(\lambda_{-i} - \hat{\lambda}_{-i}) + \eta_3 = 0, \tag{12}$$

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) - rl \geq 0, \tag{13}$$

$$u_i \geq 0, \tag{14}$$

$$\bar{v}_{i,i} \geq 0, \tag{15}$$

$$\bar{v}_{i,-i} \geq 0, \tag{16}$$

$$\gamma(\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) - rl) = 0, \tag{17}$$

$$\begin{aligned}\eta_1 u_i &= 0, & (18)\\ \eta_2 \bar{v}_{i,i} &= 0, & (19)\\ \eta_3 \bar{v}_{i,-i} &= 0, & (20)\end{aligned}$$

where γ, η_1, η_2, and η_3 are Lagrangian multipliers and $\gamma, \eta_1, \eta_2, \eta_3 \leq 0$. Equation (10)-(12) are first-order conditions, (13) is the NIC condition, and inequality (14)-(16) imply that utility flow and continuation utility should be nonnegative.

To do the phase-diagram analysis, we need to determine the dynamics of v_i and $C'_{i,n}(v_i)$ at any point in the v_i-$C'_{i,n}(v_i)$ plane, which are determined by the sign of $dC'_{i,n}(v_i)/dt$ and dv_i/dt. The dynamics of v_i is given by

$$\frac{dv_i}{dt} = rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i).$$

Using the envelope theorem, we can derive the expression for $dC'_{i,n}(v_i)/dt$ from the HJB equation, which is

$$\frac{dC'_{i,n}(v_i)}{dt} = \gamma(\lambda - \hat{\lambda}_{-i}).$$

Therefore, given a point in the v_i-$C'_{i,n}(v_i)$ plane, the values of $(u_i, \bar{v}_{i,i}, \bar{v}_{i,-i}, \gamma)$ can be derived from Kuhn-Tucker conditions, which in turn determine the values of dv_i/dt and $dC'_{i,n}(v_i)/dt$. Finally, the sign of dv_i/dt and $dC'_{i,n}(v_i)/dt$ determine the dynamics at this point. The following seven lemmas analyze the dynamics.

Lemma B.1 *If $C'_{i,n}(v_i) \geq C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$, then the NIC condition is non-binding. The dynamics of $C'_{i,n}(v_i)$ and v_i satisfy*

$$\begin{aligned}\frac{dC'_{i,n}(v_i)}{dt} &= 0,\\ \frac{dv_i}{dt} &< 0.\end{aligned}$$

Proof of Lemma B.1: We prove the first part by guess-and-verify method. Suppose that the NIC condition is not binding and both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are strictly positive. It follows that all the Lagrangian multipliers γ, η_2 and η_3 equal to 0. Then, first-order conditions (11) and (12) imply that $C'_{i,n+1}(\bar{v}_{i,i}) = C'_{i,n+1}(\bar{v}_{i,-i}) = C'_{i,n}(v_i)$. Because $C'_{i,n}(v_i) \geq C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ and $C'_{i,n+1}(v_i)$ is strictly increasing, it follows that $\bar{v}_{i,i} = \bar{v}_{i,-i} \geq v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}$. Hence,

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) \geq (\lambda - \hat{\lambda}_{-i})\frac{rl}{\lambda - \hat{\lambda}_{-i}} = rl.$$

Thus, the NIC condition is non-binding, and both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are strictly positive, which verifies our guess. Then, the multiplier γ equals 0 and the dynamics of $C'_{i,n}(v_i)$ satisfies

$$\frac{dC'_{i,n}(v_i)}{dt} = \gamma(\lambda - \hat{\lambda}_{-i}) = 0.$$

Next, we analyze the dynamics of v_i under two cases.

(i) $\lambda_{-i} \leq \hat{\lambda}_{-i}$

Because $C'_{i,n+1}(v_i) \geq S'(v_i)$ for all v_i by Assumption B, it follows that

$$C'_{i,n}(v_i) \geq C'_{i,n+1}\left(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}\right) \geq S'\left(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}\right) > S'(v_i).$$

When $C'_{i,n}(v_i) \geq C'_{i,n+1}\big(v_i + \frac{rl}{\lambda-\hat{\lambda}_{-i}}\big) > 0$, utility-flow u_i is determined by first-order condition $S'(u_i) = C'_{i,n}(v_i)$. Consequently, $S'(u_i) > S'(v_i)$, which implies that $u_i > v_i$. Finally, we have

$$\begin{aligned}
\frac{dv_i}{dt} &= rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i) \\
&\leq r(v_i - u_i) - \frac{rl\hat{\lambda}_{-i}}{\lambda - \hat{\lambda}_{-i}} \\
&< 0,
\end{aligned}$$

where the first inequality follows from $\bar{v}_{i,i} = \bar{v}_{i,-i} \geq v_i + \frac{rl}{\lambda-\hat{\lambda}_{-i}}$, and the second inequality follows from $u_i > v_i$.

(ii) $\lambda_{-i} > \hat{\lambda}_{-i}$

When $v_i \geq \frac{\hat{\lambda}_{-i}l}{\lambda-\hat{\lambda}_{-i}}$, we have

$$S'(u_i) = C'_{i,n}(v_i) \geq C'_{i,n+1}\left(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}\right) > C'_{i,n+1}(v_i) > S'\left(v_i - \frac{\hat{\lambda}_{-i}l}{\lambda - \hat{\lambda}_{-i}}\right),$$

where the last inequality follows from Assumption B. Hence, $u_i > v_i - \frac{\hat{\lambda}_{-i}l}{\lambda-\hat{\lambda}_{-i}}$. When $v_i < \frac{\hat{\lambda}_{-i}l}{\lambda-\hat{\lambda}_{-i}}$, we have $u_i \geq 0 > v_i - \frac{\hat{\lambda}_{-i}l}{\lambda-\hat{\lambda}_{-i}}$. Thus, we always have $u_i > v_i - \frac{\hat{\lambda}_{-i}l}{\lambda-\hat{\lambda}_{-i}}$. Therefore,

$$\begin{aligned}
\frac{dv_i}{dt} &= rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i) \\
&\leq r(v_i - u_i) - \frac{r\hat{\lambda}_{-i}l}{\lambda - \hat{\lambda}_{-i}} \\
&< \frac{r\hat{\lambda}_{-i}l}{\lambda - \hat{\lambda}_{-i}} - \frac{r\hat{\lambda}_{-i}l}{\lambda - \hat{\lambda}_{-i}} \\
&= 0.
\end{aligned}$$

Q.E.D.

Lemma B.2 *If $C'_{i,n}(v_i) < C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$, then the NIC condition is binding. The dynamics $C'_{i,n}(v_i)$ satisfies*

$$\frac{dC'_{i,n}(v_i)}{dt} < 0.$$

PROOF OF LEMMA B.2: On the contrary, suppose the NIC condition is non-binding and hence $\gamma = 0$. The first-order conditions (11) and (12) imply that

$$\bar{v}_{i,i} = \bar{v}_{i,-i} < v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}},$$

where the last inequality follows from the condition that $C'_{i,n}(v_i) < C'_{i,n+1}\big(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}\big)$. Then,

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) < (\lambda - \hat{\lambda}_{-i})\frac{rl}{\lambda - \hat{\lambda}_{-i}} = rl.$$

The NIC condition is violated, which is a contradiction. Therefore, we must have the NIC condition binds and $\gamma < 0$. Then, the dynamics $C'_{i,n}(v_i)$ satisfies

$$\frac{dC'_{i,n}(v_i)}{dt} = \gamma(\lambda - \hat{\lambda}_{-i}) < 0.$$

Q.E.D.

To analyze the dynamics of v_i, we fix the value of v_i and variate the value of $C'_{i,n}(v_i)$. When we do this, the value of dv_i/dt could be treated as a function of the value of $C'_{i,n}(v_i)$. Denote this function by $f(C'_{i,n}(v_i))$, the sign of which determined the dynamics of v_i. When the NIC condition is binding, we have

$$f(C'_{i,n}(v_i)) = r(v_i - u_i) - \hat{\lambda}_{-i}(\bar{v}_{i,-i} - v_i).$$

Given $C'_{i,n}(v_i)$, $\{u_i, \bar{v}_{i,i}, \bar{v}_{i,-i}\}$ are determined by the system of Kuhn-Tucker conditions. Moreover, since both S' and $C'_{i,n+1}$ are continuous functions, it follows that $f(C'_{i,n}(v_i))$ is continuous in $C'_{i,n}(v_i)$. Moreover, we have

Lemma B.3 *Fixing v_i, $f(C'_{i,n}(v_i))$ is a decreasing continuous function of $C'_{i,n}(v_i)$.*

PROOF OF LEMMA B.3: We prove this lemma under three cases.

Case 1: $\lambda_{-i} = \hat{\lambda}_{-i}$

When $C'_{i,n}(v_i) \geq 0$, u_i and $\bar{v}_{i,-i}$ are determined by the following first-order conditions

$$\begin{aligned} S'(u_i) - C'_{i,n}(v_i) &= 0, \\ C'_{i,n+1}(\bar{v}_{i,-i}) - C'_{i,n}(v_i) &= 0. \end{aligned}$$

Since both S' and $C'_{i,n+1}$ are strictly increasing functions, when we decrease the value of $C'_{i,n}(v_i)$, both u_i and $\bar{v}_{i,-i}$ decrease, and hence $f(C'_{i,n}(v_i))$ increases. Therefore, $f(C'_{i,n}(v_i))$ is a strictly decreasing function of $C'_{i,n}(v_i)$ when $C'_{i,n}(v_i) \geq 0$. If $C'_{i,n}(v_i) < 0$, then both u_i and $\bar{v}_{i,-i}$ equal 0. Hence, $f(C'_{i,n}(v_i))$ is a constant function of $C'_{i,n}(v_i)$ when $C'_{i,n}(v_i) < 0$.

Case 2: $\lambda_{-i} < \hat{\lambda}_{-i}$

When $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$, from the proof of Lemma B.1, both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ equal to $v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}} > 0$. Hence, both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are positive when $C'_{i,n}(v_i)$ is very close to $C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$. When both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are positive, they are determined by the following system of equations

$$\lambda_i C'_{i,n+1}(\bar{v}_{i,i}) - \lambda_i C'_{i,n}(v_i) + \gamma\lambda_i = 0, \tag{21}$$

$$\lambda_{-i} C'_{i,n+1}(\bar{v}_{i,-i}) - \lambda_{-i} C'_{i,n}(v_i) + \gamma(\lambda_{-i} - \hat{\lambda}_{-i}) = 0, \tag{22}$$

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) = rl. \tag{23}$$

Since $\lambda_i > 0$, $\lambda_{-i} - \hat{\lambda}_{-i} < 0$, and $\gamma < 0$, (21) and (22) imply that

$$C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n}(v_i) > C'_{i,n+1}(\bar{v}_{i,-i}).$$

Combining (21) and (22), we have

$$(\lambda_{-i} - \hat{\lambda}_{-i}) C'_{i,n+1}(\bar{v}_{i,i}) - \lambda_{-i} C'_{i,n+1}(\bar{v}_{i,-i}) = -\hat{\lambda}_{-i} C'_{i,n}(v_i). \tag{24}$$

When we decrease the value of $C'_{i,n}(v_i)$ starting from $C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$, the right-hand side of (24) increases. It follows from (23) that $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ variate in the same direction. Since $C'_{i,n+1}(\cdot)$ is an increasing function, to let (24) hold, we must have both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ decrease when $C'_{i,n}(v_i)$ decreases. Moreover, we know that u_i also decreases when $C'_{i,n}(v_i)$ goes down. Hence, $f(C'_{i,n}(v_i))$ increases when $C'_{i,n}(v_i)$ decreases, as long as both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are positive.

When we keep decreasing $C'_{i,n}(v_i)$, $\bar{v}_{i,-i}$ may hit the lower bound 0. Then, if we decrease $C'_{i,n}(v_i)$ further, $\bar{v}_{i,-i}$ remains at 0 and u_i continues to decrease as long as $C'_{i,n}(v) \geq 0$. Thus, $f(C'_{i,n}(v_i))$ still increases as $C'_{i,n}(v_i)$ decreases until $C'_{i,n}(v_i)$ reaches 0. Finally, when $C'_{i,n}(v_i) < 0$, both u_i and $\bar{v}_{i,-i}$ equal 0, and $f(C'_{i,n}(v_i))$ becomes constant function of $C'_{i,n}(v_i)$.

Case 3: $\lambda_{-i} > \hat{\lambda}_{-i}$

Similar to the previous case, we decrease $C'_{i,n}(v_i)$ starting from $C'_{i,n+1}(v + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$. When both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are positive, they are determined by the following system of equations (21)-(23).

Since $\lambda_i > 0$, $\lambda_{-i} - \hat{\lambda}_{-i} > 0$ and $\gamma < 0$, (21) and (22) imply that

$$C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n+1}(\bar{v}_{i,-i}) > C'_{i,n}(v_i).$$

When $C'_{i,n}(v_i)$ decreases, the right-hand side of (24) goes up. Different from the previous case, (23) implies that $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ variate in different direction. Thus, to let (24) hold, we must have $\bar{v}_{i,i}$ increase and $\bar{v}_{i,-i}$ decrease. Moreover, u_i decreases when $C'_{i,n}(v_i)$ decreases. Hence, $f(C'_{i,n}(v_i))$ increases as we decrease the value of $C'_{i,n}(v_i)$, as long as both $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$ are positive.

When $C'_{i,n}(v_i)$ becomes non-positive, utility flow u_i remains at 0. But $\bar{v}_{i,-i}$ keeps going down as we continue to decrease $C'_{i,n}(v_i)$. Thus, $f(C'_{i,n}(v_i))$ keeps increasing as the value of $C'_{i,n}(v_i)$ decreases. Finally, $\bar{v}_{i,-i}$ hits the lower bound 0. Denote the value of $C'_{i,n}(v_i)$ at which $\bar{v}_{i,-i}$ reaches 0 at the first time by $\tilde{C}'_{i,n}(v_i)$ (we will use it in the proof of Lemma B.7). From then on, both $\bar{v}_{i,-i}$ and u_i remain at 0 as we keep decreasing the value of $C'_{i,n}(v_i)$, and therefore $f(C'_{i,n}(v_i))$ becomes a constant function of $C'_{i,n}(v_i)$.

To summarize, in all of the above three cases, $f(C'_{i,n}(v_i))$ is a continuous and decreasing function of $C'_{i,n}(v_i)$. *Q.E.D.*

For the case in which $\lambda_{-i} \leq \hat{\lambda}_{-i}$, we have

Lemma B.4 *If $C'_{i,n}(v_i) = S'(v_i)$, then $dv_i/dt \geq 0$ when $v_i > 0$, and $dv_i/dt = 0$ when $v_i = 0$.*

Proof of Lemma B.4: If $C'_{i,n}(v_i) = S'(v_i)$, then $C'_{i,n}(v_i) < C'_{i,n+1}\big(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}\big)$, because $S'(v_i) \leq C'_{i,n+1}(v_i) < C'_{i,n+1}\big(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}}\big)$ by Assumption B. Then, Lemma B.2 implies that the NIC condition is binding. Therefore,

$$\frac{dv_i}{dt} = r(v_i - u_i) - \hat{\lambda}_{-i}(\bar{v}_{i,-i} - v_i).$$

Utility flow u_i satisfies the first-order condition that $S'(u_i) = C'_{i,n}(v_i)$. Then, $S'(u_i) = C'_{i,n}(v_i) = S'(v_i)$ implies that $u_i = v_i$. In the proof of Lemma B.3, we have shown that $C'_{i,n+1}(\bar{v}_{i,-i}) \leq C'_{i,n}(v_i)$ when $\lambda_{-i} \leq \hat{\lambda}_{-i}$. Consequently,

$$S'(\bar{v}_{i,-i}) \leq C'_{i,n+1}(\bar{v}_{i,-i}) \leq C'_{i,n}(v_i) = S'(v_i),$$

which implies that $\bar{v}_{i,-i} \leq v_i$. Finally, combining $u_i = v_i$ and $\bar{v}_{i,-i} \leq v_i$, we get $dv_i/dt \geq 0$.

When $v_i = 0$, $C'_{i,n}(0) = S'(0) = 0$, which implies that $u_i = \bar{v}_{i,-i} = 0$. Thus, $dv_i/dt = 0$ when $v_i = 0$. *Q.E.D.*

Lemma B.5 *The $dv_i/dt = 0$ locus is a continuous curve that locates below the $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ locus and above the $C'_{i,n}(v_i) = S'(v_i)$ locus and intersects the $C'_{i,n}(v_i) = S'(v_i)$ locus at the origin.*

Proof of Lemma B.5: By Lemma B.1, $dv_i/dt < 0$ on the $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ locus. By Lemma B.4, $dv_i/dt \geq 0$ on the $C'_{i,n}(v_i) = S'(v_i)$ locus, with strict inequality when $v_i > 0$. Moreover, Lemma B.3 shows that, fixing v_i, the value of dv_i/dt is a continuous and strictly decreasing function of $C'_{i,n}(v_i)$ when $C'_{i,n}(v_i) \geq 0$. Therefore, for any $v_i \geq 0$, there exists an unique value of $C'_{i,n}(v_i)$ between $S'(v_i)$ and $C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ such that $dv_i/dt = 0$. Moreover, the $dv_i/dt = 0$ locus is determined by the system of Kuhn-Tucker conditions and the following condition

$$rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i) = 0,$$

and both S', $C'_{i,n+1}$ are continuous functions. Therefore, the $dv_i/dt = 0$ locus is a continuous curve that locates below the $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ locus and above the $C'_{i,n}(v_i) = S'(v_i)$ locus. Finally, $dv_i/dt = 0$ at $C'_{i,n}(0) = S'(0) = 0$ by Lemma B.4. Thus, the $dv_i/dt = 0$ locus intersects the $C'_{i,n}(v_i) = S'(v_i)$ locus at the origin. *Q.E.D.*

For the case in which $\lambda_{-i} > \hat{\lambda}_{-i}$, we have

Lemma B.6 *If $C'_{i,n}(v_i) = S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$, then $dv_i/dt > 0$.*

Proof of Lemma B.6: If $C'_{i,n}(v_i) = S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$, then $C'_{i,n}(v_i) < C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$, because $S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}) < C'_{i,n+1}(v_i) < C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ by Assumption B. Then, Lemma B.2 implies that the NIC condition is binding. Thus,

$$\lambda_i(\bar{v}_{i,i} - v_i) + (\lambda_{-i} - \hat{\lambda}_{-i})(\bar{v}_{i,-i} - v_i) - rl = 0. \tag{25}$$

In the proof of Lemma B.3, we have shown that $C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n+1}(\bar{v}_{i,-i})$, and hence $\bar{v}_{i,i} > \bar{v}_{i,-i}$. Then it follows from (25) that

$$\bar{v}_{i,i} > v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}} > \bar{v}_{i,-i}.$$

Utility flow u_i is determined by the first-order condition $S'(u_i) = C'_{i,n}(v_i)$, which implies $u_i =$

$v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$. It follows that

$$\begin{aligned} \frac{dv_i}{dt} &= r(v_i - u_i) - \hat{\lambda}_{-i}(\bar{v}_{i,-i} - v_i) \\ &= \frac{r\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}} - \hat{\lambda}_{-i}(\bar{v}_{i,-i} - v_i) \\ &> \frac{r\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}} - \frac{r\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}} \\ &= 0. \end{aligned}$$

Q.E.D.

Lemma B.7 *The $dv_i/dt = 0$ locus is a continuous curve that locates below the $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ locus and above the $C'_{i,n}(v_i) = S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$ locus.*

Proof of Lemma B.7: When $v_i \geq \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$, $dv_i/dt < 0$ on the $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ locus by Lemma B.1, and $dv_i/dt > 0$ on the $C'_{i,n}(v_i) = S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$ locus by Lemma B.6. Moreover, fixing v_i, Lemma B.3 shows that the value of dv_i/dt is a continuous and strictly decreasing function of $C'_{i,n}(v_i)$ when $C'_{i,n}(v_i) \geq 0$. Therefore, there exists an unique value of $C'_{i,n}(v_i)$, which is between $C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ and $S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$, such that $dv_i/dt = 0$.

Next, we consider the case when $0 \leq v_i < \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$. From the proof of Lemma B.3, when $C'_{i,n}(v_i)$ equals $\tilde{C}'_{i,n}(v_i)$, we have $u_i = \bar{v}_{i,-i} = 0$ and hence $dv_i/dt = r(v_i - u_i) - \hat{\lambda}_{-i}(\bar{v}_{i,-i} - v_i) \geq 0$. Moreover, the value of dv_i/dt is a continuous and strictly decreasing function of $C'_{i,n}(v_i)$ when $C'_{i,n}(v_i) \geq \tilde{C}'_{i,n}(v_i)$. Therefore, there exists an unique value of $C'_{i,n}(v_i)$, which is between $C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ and $\tilde{C}'_{i,n}(v_i)$, such that $dv_i/dt = 0$.

Thus, the $dv_i/dt = 0$ locus locates below the $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ locus and the $C'_{i,n}(v_i) = S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$ locus. Furthermore, the $dv_i/dt = 0$ locus is a continuous curve by the same argument as in the proof of Lemma B.6. *Q.E.D.*

These lemmas characterize the dynamics of v_i and $C'_{i,n}(v_i)$ in the v_i-$C'_{i,n}(v_i)$ plane. The $dv_i/dt = 0$ locus determines the dynamics of v_i: v_i is decreasing over time above it and increasing over time below it. The $C'_{i,n}(v_i) = C'_{i,n+1}(v_i + \frac{rl}{\lambda - \hat{\lambda}_{-i}})$ locus determines the dynamics of $C'_{i,n}(v_i)$: $C'_{i,n}(v_i)$ is constant over time above it and decreasing over time below it (Figure 4 and Figure 5).

The next step is to find the optimal path in these phase diagrams. First consider the phase diagram for the case in which $\lambda_{-i} \leq \hat{\lambda}_{-i}$ (Figure 4). From the theorem regarding the existence of a solution to a differential equation, there is an unique path from any $v_i > 0$ to the origin (Path 1 in Figure 4). First, any path on which the state variable v_i diverges to infinity could be ruled out. This contains the area below Path 1. In the area above Path 1, the continuation-utility v_i is decreasing over time. When v_i hits the lower bound 0, it cannot decrease any further. Thus, we must have $dv_i/dt \geq 0$ at $v_i = 0$. This condition rules out any path above Path 1, because $dv_i/dt < 0$ at $v_i = 0$ on these paths. Then, Path 1 is the only candidate path left in the phase diagram, and hence it is the optimal path that we are looking for. The final step is to pin down the boundary condition at $v_i = 0$. When $v_i = 0$, we have $u_i = \bar{v}_{i,-i} = 0$ and $\bar{v}_{i,i} = \frac{rl}{\lambda_i}$. Then,

$$\frac{dv_i}{dt} = rv_i - r(u_i - l) - \lambda_i(\bar{v}_{i,i} - v_i) - \lambda_{-i}(\bar{v}_{i,-i} - v_i) = 0.$$

Therefore, when agent i's continuation-utility reaches 0, his continuation-utility and transferred-utility-flow remain at 0 until he makes a discovery. To force agent i to put in positive effort, the principal rewards him by increasing his continuation-utility to $\frac{rl}{\lambda_i}$ when he makes a discovery. We also can pin down the following boundary condition at $v_i = 0$ from the HJB equation

$$C_{i,n}(0) = \frac{\lambda_i C_{i,n+1}(\frac{rl}{\lambda_i}) + \lambda_{-i} C_{i,n+1}(0)}{r + \lambda}.$$

The optimal path and the boundary condition together determine the solution of the HJB equation. The phase-diagram analysis for the case in which $\lambda_{-i} > \hat{\lambda}_{-i}$ is similar (Figure 5).

Finally, from the phase-diagram, when $\lambda_{-i} \leq \hat{\lambda}_{-i}$, the optimal path is located above the $C'_{i,n}(v_i) = S'(v_i)$ locus and intersects the $C'_{i,n}(v_i) = S'(v_i)$ locus at the origin; when $\lambda_{-i} > \hat{\lambda}_{-i}$, the optimal path is located above the $C'_{i,n}(v_i) = S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$ locus. Therefore, we have the following Lemma.

Lemma B.8 *(i) If $\lambda_{-i} \leq \hat{\lambda}_{-i}$, then $C'_{i,n}(v_i) \geq S'(v_i)$ for all $v_i > 0$, and $C'_{i,n}(0) = S'(0) = 0$.*

(ii) If $\lambda_{-i} > \hat{\lambda}_{-i}$, then $C'_{i,n}(v_i) > S'(v_i - \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}})$ for all $v_i \geq \frac{\hat{\lambda}_{-i} l}{\lambda - \hat{\lambda}_{-i}}$.

Lemma B.8 indicates that, if $C_{i,n+1}$ satisfies Assumption B, then $C_{i,n}$ also does. This result completes the final step of the backward-induction argument.

Proposition 4.2 summarizes the properties of the optimal dynamic contract for agent i. We provide the proof below.

Proof of Proposition 4.2: For part (i), it has been shown that $C_{i,n}(v_i)$ is determined by the HJB equation and the boundary condition. From the phase diagram, $C'_{i,n}(v_i)$ is a continuous and strictly increasing function of v_i. It follows that $C_{i,n}(v_i)$ is a convex function.

To describe the dynamics of transferred-utility flow, let u_i, $\bar{u}_{i,i}$ and $\bar{u}_{i,-i}$ be the corresponding utility-flow when the continuation utility are v_i, $\bar{v}_{i,i}$ and $\bar{v}_{i,-i}$.

When all of $C'_{i,n}(v_i)$, $C'_{i,n+1}(\bar{v}_{i,i})$ and $C'_{i,n+1}(\bar{v}_{i,-i})$ are positive, $(u_i, \bar{u}_{i,i}, \bar{u}_{i,-i})$ are determined by the following first-order condition

$$\begin{aligned}
S'(u_i) &= C'_{i,n}(v_i), \\
S'(\bar{u}_{i,i}) &= C'_{i,n+1}(\bar{v}_{i,i}), \\
S'(\bar{u}_{i,i}) &= C'_{i,n+1}(\bar{v}_{i,-i}).
\end{aligned}$$

If $\lambda_{-i} = \hat{\lambda}_{-i}$, (11) and (12) imply that

$$C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n}(v_i) = C'_{i,n+1}(\bar{v}_{i,-i}) \geq 0.$$

It follows that $\bar{u}_{i,i} > u_i = \bar{u}_{i,-i}$.

If $\lambda_{-i} < \hat{\lambda}_{-i}$, (11) and (12) imply that

$$C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n}(v_i) \geq C'_{i,n+1}(\bar{v}_{i,-i}) \geq 0,$$

where the second inequity is strict when $v_i > 0$. Hence, $\bar{u}_{i,i} > u_i \geq \bar{u}_{i,-i}$, with strict inequality when $v_i > 0$.

If $\lambda_{-i} > \hat{\lambda}_{-i}$, (11) and (12) imply that

$$C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n+1}(\bar{v}_{i,-i}) > C'_{i,n}(v_i).$$

Therefore, when $C'_{i,n}(v_i) \geq 0$, we have $\bar{u}_{i,i} > \bar{u}_{i,-i} > u_i$. However, derivative of the cost function could be negative when $\lambda_{-i} > \hat{\lambda}_{-i}$. In this case, the utility flow equal 0. Therefore,

- if $C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n+1}(\bar{v}_{i,-i}) > 0 \geq C'_{i,n}(v_i)$, we have $\bar{u}_{i,i} > \bar{u}_{i,-i} > u_i = 0$.
- If $C'_{i,n+1}(\bar{v}_{i,i}) > 0 \geq C'_{i,n+1}(\bar{v}_{i,-i})) > C'_{i,n}(v_i)$, we have $\bar{u}_{i,i} > 0 = \bar{u}_{i,-i} = u_i$.
- If $0 \geq C'_{i,n+1}(\bar{v}_{i,i}) > C'_{i,n+1}(\bar{v}_{i,-i}) > C'_{i,n}(v_i)$, we have $\bar{u}_{i,i} = \bar{u}_{i,-i} = u_i = 0$.

To summarize, if agent i completes the innovation, the principal rewards him by an upward jump in utility flow. If his coworker completes the innovation, then: 1) his utility-flow does not change if $\lambda_{-i} = \hat{\lambda}_{-i}$; 2) his utility-flow drops down if $\lambda_{-i} < \hat{\lambda}_{-i}$; 3) his utility-flow jumps up if $\lambda_{-i} > \hat{\lambda}_{-i}$. These results prove part (ii) and part (iii).

Finally, for part (iv), note that on the optimal path v_i is decreasing over time and asymptotically converges to 0. Moreover, the transferred utility satisfies $S'(u_i) = C'_{i,n}(v_i)$ and both S and $C_{i,n}$ are convex functions. Therefore, transferred utility u has the same dynamics as continuation utility in case of failure. *Q.E.D.*

C Proofs for Implementation

Proof of Proposition 6.1:

We first verify that $V_n(y) = C_n^{-1}(y)$ solves the HJB equation under the conditions stated in Proposition 6.1. Since C_n is a strictly increasing and differentiable function, C_n^{-1} exists and is also differentiable. Let $V_n(y) = C_n^{-1}(y)$. Then we have

$$\begin{aligned}
\lambda(V_{n+1}(Y_{n+1}(y)) - V_n(y)) - rl &= \lambda(V_{n+1}(C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})) - V_n(y)) - rl \\
&= \lambda(C_n^{-1}(y) + \frac{rl}{\lambda} - C_n^{-1}(y)) - rl \\
&= 0.
\end{aligned}$$

This result implies that suppose $V_n(y) = C_n^{-1}(y)$ is the solution then the agent is always indifferent between exerting effort and shirking, no matter how much he decides to consume. Thus, we have

$$\begin{aligned}
RHS &= rU(c^*) + V_n'(y)\dot{y} \\
&= rU(c^*) + V_n'(y)(ry - rc^* - \lambda(C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda}) - y)) \\
&= rU(c^*) + \frac{(r+\lambda)y - rc^* - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}{C_n'(C_n^{-1}(y))},
\end{aligned}$$

where c^* is the optimal choice of consumption which is determined by the first-order condition $U'(c^*) = \frac{1}{C_n'(C_n^{-1}(y))}$.

Since, from principal's problem, $C_n(v)$ satisfies the following differential equation

$$(r+\lambda)C_n(v) = rS(u^*) + C_n'(v)(r(v-u^*)) + \lambda C_{n+1}(v + \frac{rl}{\lambda}),$$

then

$$\frac{1}{C_n'(v)} = \frac{r(v-u^*)}{(r+\lambda)C_n(v) - rS(u^*) - \lambda C_{n+1}(v + \frac{rl}{\lambda})},$$

where u^* is the optimal choice of utility flow which satisfies $S'(u^*) = C_n'(v)$. Taking $v = C_n^{-1}(y)$ into the equation above, we get

$$\begin{aligned} \frac{1}{C_n'(C_n^{-1}(y))} &= \frac{r(C_n^{-1}(y) - u^*)}{(r+\lambda)C_n(C_n^{-1}(y)) - rS(u^*) - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})} \\ &= \frac{r(C_n^{-1}(y) - u^*)}{(r+\lambda)y - rS(u^*) - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}, \end{aligned}$$

where $S'(u^*) = C_n'(C_n^{-1}(y))$. Since $S(u^*) = U^{-1}(u^*)$, it follows that $\frac{1}{U'(S(u^*))} = C_n'(C_n^{-1}(y))$. Because c^* satisfies $U'(c^*) = \frac{1}{C_n'(C_n^{-1}(y))}$, we have $U'(S(u^*)) = U'(c^*)$, and hence $S(u^*) = c^*$ and $u^* = U(c^*)$. Therefore,

$$\frac{1}{C_n'(C_n^{-1}(y))} = \frac{r(C_n^{-1}(y) - U(c^*))}{(r+\lambda)y - rc^* - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}.$$

Taking this expression for $\frac{1}{C_n'(C_n^{-1}(y))}$ into the right-hand side of the HJB equation, we have

$$\begin{aligned} RHS &= rU(c^*) + \frac{(r+\lambda)y - rc^* - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}{C_n'(C_n^{-1}(y))} \\ &= rU(c^*) + r(C_n^{-1}(y) - U(c^*)) \\ &= rC_n^{-1}(y) \\ &= rV_n(y) \\ &= LHS. \end{aligned}$$

Thus, $V_n(y) = C_n^{-1}(y)$ solves the following HJB equation.

Next, we show that this implementation generates the same consumption allocation as the optimal contract. The proof above shows that the maximum expected utility that the agent can derive from the implementation for a given wealth level y is $V_n(y)$ and he is always willing to exert effort. $V_n(y)$ could also be thought as his"continuation utility" given wealth y. From the agent's HJB equation, his "continuation utility" evolves according to $V_n'(y)\dot{y} = rV_n(y) - rU(c^*)$, where c^* is the optimal choice of consumption given y. In the principal's problem, given continuation-utility v, the continuation utility evolves according to $\dot{v} = rv - ru^*$, where u^* is the optimal choice of utility flow. From the previous proof, if $v = V_n(y)$ then $u^* = U(c^*)$. Therefore, given the same continuation utility, the implementation and the optimal contract choose the same consumption, which further induces the same law of motion of continuation utility. Finally, the initial condition that $y^0 = C_1(v^0)$ grantees that the agent starts with initial continuation-utility v^0. Thus, the implementation and the optimal contract generate the same consumption allocation under all possible realization of the innovation process.

Q.E.D.

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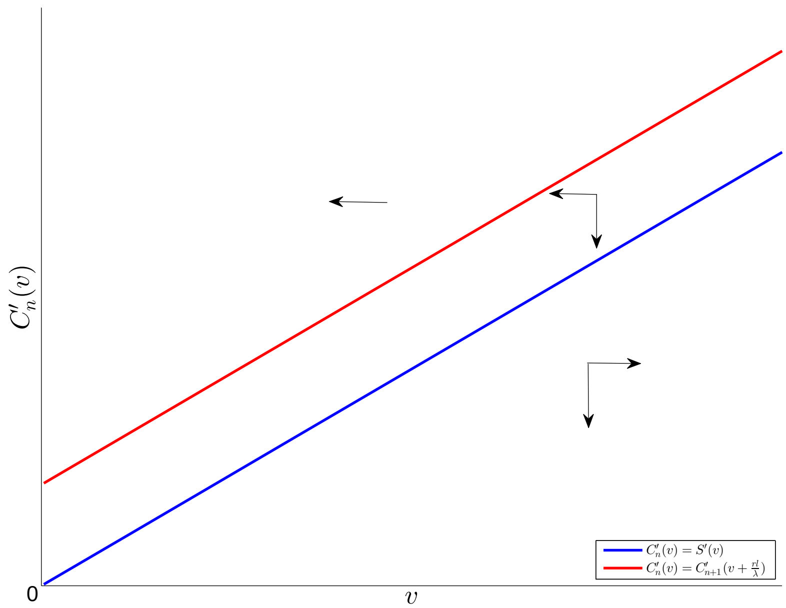

Figure 1: Phase Diagram

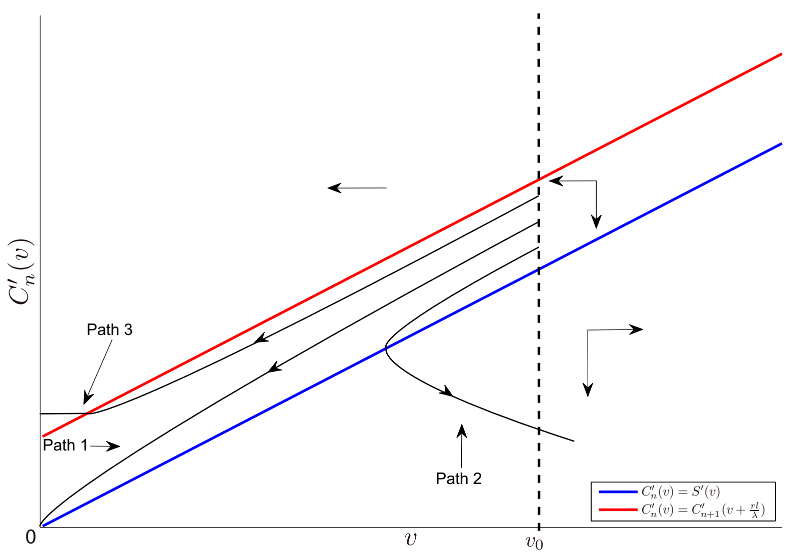

Figure 2: Optimal Path

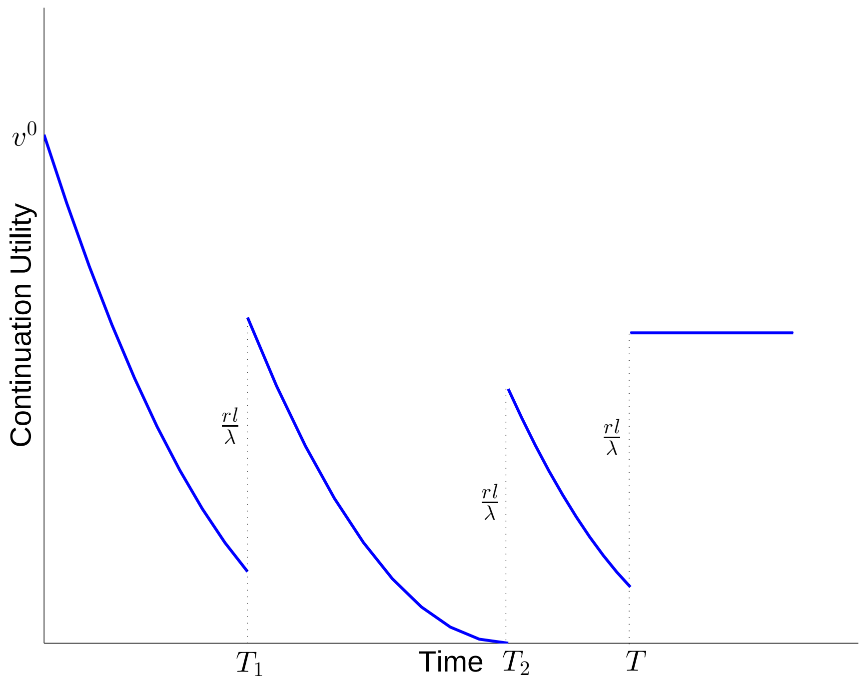

Figure 3: Multi-stage

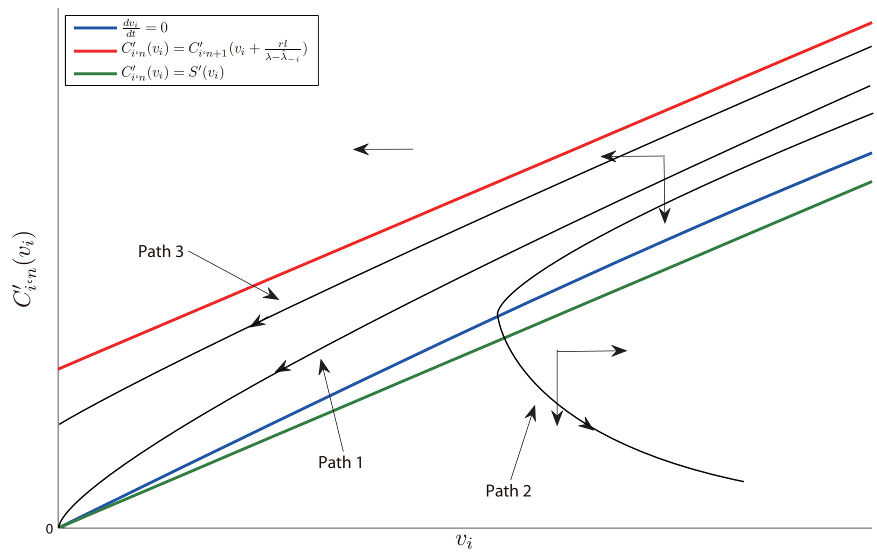

Figure 4: Phase Diagram ($\lambda_{-i} \leq \hat{\lambda}_{-i}$)

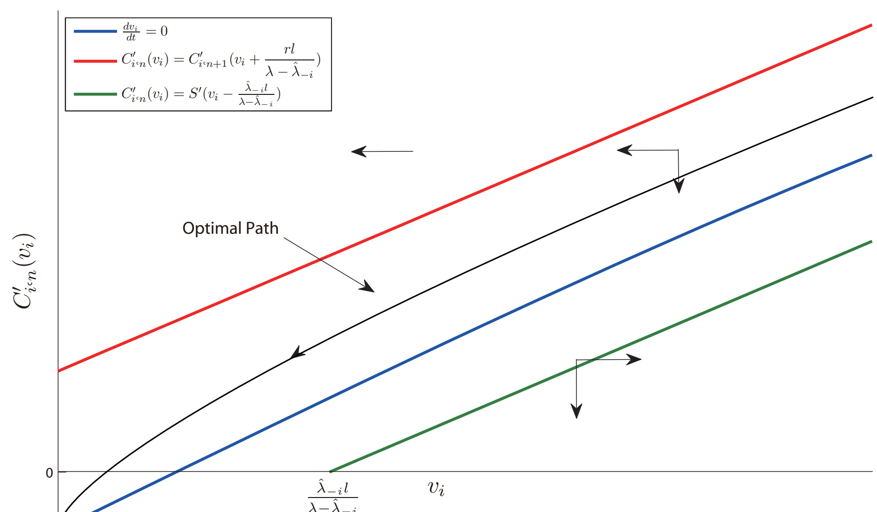

Figure 5: Phase Diagram ($\lambda_{-i} > \hat{\lambda}_{-i}$)