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Policy Polarization and Strategic Candidacy in Elections under the Alternative Vote Rule

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We use the citizen-candidate model to study electoral outcomes under the Alternative Vote rule, a voting method often proposed as a replacement to the prevalent Plurality rule. We show that, like the Plurality rule, the Alternative Vote rule deters multiple candidate clusters and the presence of candidates at more than two positions. Moreover, the Alternative Vote rule tends to support less policy polarization than the Plurality rule. These results stand in contrast to those obtained under other proposed voting rules, Approval Voting in particular, which are prone to candidate clustering and, as a result, can support greater policy polarization vis-à-vis the Plurality rule.

Key Words: Alternative Vote rule; Instant Runoff Voting; Citizen-candidate model; Endogenous candidacy; policy polarization.

Subject Classification: C72; D72; D78.

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1. INTRODUCTION

The Plurality rule¹ is a commonly used voting method in political elections across major democracies such as the US, Canada and the UK. Electoral reformers dissatisfied with the use of the Plurality rule have often considered the Alternative Vote rule as an appealing substitute to the Plurality rule. In the US, adoption of the Alternative Vote rule for political elections has been actively advocated by, among others, the Center for Voting and Democracy (fairvote.org). The Canadian province of British Columbia held in 2009 a referendum on the adoption of the Alternative Vote rule for provincial elections. More recently, the adoption of the Alternative Vote rule for national elections has been advocated by the Liberal Party of Canada and in May 2011 the UK held a referendum on the adoption of the Alternative Vote rule for elections to the House of Commons. The Alternative Vote rule has been in use for elections to the Lower House in Australia since 1918, and variants of the Alternative Vote rule have been adopted lately for local elections in several large cities (e.g., London, San Francisco).²

The Alternative Vote rule works as follows: Each voter rank-orders candidates from first to last. A candidate who receives a majority of first place preferences is declared the winner. If no candidate receives a majority of first place preferences, then the candidate i getting the *least* number of first place preferences is eliminated and the candidates who are ranked second after candidate i become first on the ballots. If a candidate receives a majority of first place preferences on the altered ballots, then he is elected. Otherwise, the process is repeated until a candidate receives a majority of first place preferences.³

An argument sometimes put forward in favor of the Alternative Vote rule vis-à-vis the Plurality rule is that the Alternative Vote rule provides better electoral prospects for centrist/moderate candidates over extremists which, under standard assumptions on the distribution of voters' policy preferences, are associated with an

¹The Plurality rule is the voting rule under which each voter casts one vote to one candidate, and the candidate who receives the most votes is elected.

²For a list of places that currently use the Alternative Vote rule for elections, see fairvote.org.

³The Alternative Vote rule is variously referred to as Instant-Runoff Voting (IRV), Transferable Vote, Ranked Choice Voting, or Preferential Voting. There also exists variants of the basic system, for instance, the Coombs Rule under which the candidate receiving the most last place preferences is eliminated.

increase in social welfare as measured, for example, by a utilitarian or a Rawlsian social welfare function. For instance, Merrill (1988) and Grofman and Feld (2004), focusing on the mechanical aspects of the voting rules, show that the Alternative Vote rule is more likely to elect the Condorcet winner (if one exists) than the Plurality rule does. In a standard spatial voting setting with a single-dimensional policy space and single-peaked preferences, the Condorcet winner is a centrist/moderate candidate in the sense that it is the candidate most preferred by the median voter. A similar type of conclusion can be reached if one focuses instead on candidates' platform choices. In the standard Downsian framework, where a fixed number of candidates choose their platforms along the line in order to maximize their respective probabilities of being elected and where voters have single-peaked preferences, there exists a single convergent equilibrium⁴ under the Alternative Vote rule. Under this equilibrium, all candidates choose to stand at the median voter's ideal policy.⁵ The extent of policy polarization is therefore minimal. By contrast, Cox (1987, 1990) shows that no convergent equilibria exist in multi-candidate elections under the Plurality rule. Thus, the Plurality rule provides incentives for candidates to diverge, while the Alternative Vote rule provides incentives for candidates to converge.

One restrictive feature of the existing literature on the comparative properties of the Plurality and Alternative Vote rules is that the analysis is carried out under the assumption of a common set of candidates/alternatives. However, it is well known that different voting rules provide different incentives for candidates to stand for election. Dutta et al. (2001) shows that, under a very weak unanimity condition, every non-dictatorial voting rule is subject to strategic candidacy behavior. Lijphart (1994), among others, shows that different electoral systems are associated with different numbers of candidates/parties. It is therefore important, when comparing voting rules, to explicitly take into account the fact that different voting rules may result in systematically different sets of candidates (with different policy platforms) running for election.

In this paper, we make three contributions to the existing literature. First, we develop a formal model of electoral competition under the Alternative Vote rule

⁴A convergent equilibrium is a symmetric equilibrium, where all candidates follow the same strategy.

⁵A formal proof is available from the authors.

when candidacy decisions are strategic.⁶ This enables us to evaluate the endogenously determined number and positions of candidates under the Alternative Vote rule with those under the Plurality rule.

Second, our analysis suggests that, similar to the Plurality rule, the Alternative Vote rule tends to support a two-party system (formally, the equilibrium number of candidates does not exceed two). This finding is consistent with empirical observations from actual elections held under the Alternative Vote rule (e.g., Jesse, 2000; Farrell and McAllister, 2006). Interestingly, this happens in our framework because of strategic candidacy behavior, a channel different from the standard explanation for Duverger’s law⁷ which relies on strategic voting behavior. It is well known that the Plurality rule also leads to a two-party system. This finding suggests that the Alternative Vote rule, if it were to yield systematically different outcomes vis-à-vis the Plurality rule, must do so via the positions of the candidates, not the number of candidates standing for election.

Finally, we show that the policy positions supported as electoral equilibria under the Alternative Vote rule are (weakly) less polarized as compared to those under the Plurality rule, even when candidacy is endogenized. This finding is not *a priori* obvious in light of our previous findings concerning the non-ranking scoring rules (Dellis and Oak, 2014).⁸ In that paper we show that non-ranking scoring rules support less policy polarization than the Plurality rule when the set of candidates is fixed, but can support *more* policy polarization when candidacy is endogenous. This happens because, unlike the Plurality rule, non-ranking scoring rules provide incentives for multiple candidates to stand for election at the same position, which worsens the electoral prospects of centrist/moderate candidates compared to when the election is held under the Plurality rule.

The remainder of the paper is organized as follows. Section 2 discusses the related literature. Section 3 presents the model. Section 4 characterizes equilibria and compares policy polarization under the Plurality rule and the Alternative Vote

⁶We continue to assume that voting is sincere, which is consistent with experimental evidence on voting behavior in Alternative Vote rule elections (e.g., Van der Straeten et al., 2010) and with intuition given the complexity of the Alternative Vote rule.

⁷Duverger’s law (Duverger, 1954; Riker, 1982) states that the Plurality rule tends to favor a two-party system.

⁸A non-ranking scoring rule is a voting rule under which every voter is given multiple votes to cast for different candidates, and the candidate who receives the most votes is elected.

rule. Section 5 concludes with a summary of the main results and a discussion of the robustness of our results. All proofs are in the Appendix.

2. RELATED LITERATURE

Our paper contributes to the literature on endogenous candidacy in electoral competition. There are several approaches in the literature towards modeling candidate entry/exit. Some papers consider a model with two established candidates and study the threat of entry by a third candidate (e.g., Palfrey, 1984; Weber, 1992 and 1998; Callander, 2005). Others adopt a Downsian framework where purely office-motivated potential candidates decide whether to stand for election and on which platform (e.g., Feddersen et al., 1990; Osborne, 1993 and 2000; Xefteris, 2014). Still others adopt a citizen-candidate framework where policy-motivated potential candidates decide whether to stand for election, with their respective ideal policies as the only credible policy platforms (e.g., Osborne and Slivinski, 1996; Besley and Coate, 1997; Morelli, 2004; Eguia, 2007; Dellis, 2009; Brusco and Roy, 2011; Grosser and Palfrey, 2014). Our paper belongs to the citizen-candidate tradition. It adds to this literature by comparing policy polarization under the Alternative Vote rule and the Plurality rule.

Our paper also contributes to the literature that compares policy polarization under different electoral systems. This literature is again comprised of several different approaches to modeling electoral competition. Some papers adopt a Downsian approach, where a fixed set of candidates compete for office by choosing platforms, which are typically modeled as points on a line segment (e.g., Cox, 1987 and 1990; Matakos et al., 2014). Our paper uses a similar set up but assumes, as per the citizen-candidate paradigm, that candidates' ideal points are their only credible policy platforms. We then study which of the potential candidates choose to enter, and who is elected, under different voting rules. Some papers take an experimental approach to studying polarization under alternative electoral systems by focussing on voting behavior (e.g., Forsythe et al., 1996; van der Straeten et al., 2010). Our paper differs from these contributions in that we adopt a theoretical approach and focus on the candidacy behavior while assuming voting to be sincere. Our paper is most closely related, and adds to, papers that use the citizen-candidate approach to study policy polarization under different electoral systems (e.g., Osborne and

Slivinski, 1996; Morelli, 2004; Dellis and Oak, 2006 and 2014; Dellis, 2009). Our paper is the first to use this framework to study the Alternative Vote rule and compare the degree of policy polarization under it with that under the Plurality rule.

Grofman and Feld (2004) compares Condorcet efficiency (i.e., the likelihood of electing the Condorcet winner, the candidate who would defeat any other candidate in a pairwise contest) under the Plurality rule and the Alternative Vote rule. Grofman and Feld consider a one-dimensional policy space with a fixed set of candidates, single-peaked preferences and sincere voting behavior. They find that the Alternative Vote rule is at least as likely to select the Condorcet winner as the Plurality rule if there are four or less candidates, but that this result does not carry over to settings with more than four candidates.⁹ Since a moderate candidate in our setting is also the Condorcet winner, we conclude, in the similar vein as Grofman and Feld (2004), that the Alternative Vote rule is (weakly) superior to the Plurality rule in terms of Condorcet efficiency. However, while Grofman and Feld assume exogenous candidacy and focus on the mechanical effect of the voting rules, we allow for endogenous candidacy.

Finally, our paper is related to the literature on “clone candidates”. Intuitively, candidates are said to be clones if they are ranked next to each other in every voter’s preference ranking. A voting rule is said to satisfy the Independence of Clones if the election outcome does not change following the elimination of clones. Proposed by Tideman (1987), the Independence of Clones criterion is considered a desirable property of a voting rule; it suggests a form of robustness of the voting rule to strategic manipulation via candidate nominations. In our framework, clones are candidates standing at the same position. Consistent with Tideman, who finds that the Alternative Vote rule satisfies the Independence of Clones criterion, we find that the Alternative Vote rule deters clone candidates thereby satisfying the said criterion.

⁹Observe though that the example Grofman and Feld use to show that the Plurality rule can be more likely than the Alternative Vote rule to select the Condorcet winner when there are more than four candidates is not robust to endogenous candidacy.

3. MODEL

In this section we adopt a simple citizen-candidate model, based on Osborne and Slivinski (1996) and Dellis and Oak (2014), to characterize equilibrium outcomes under the Alternative Vote rule and the Plurality rule.

3.1. The environment

Consider a community that must elect a policymaker to choose and implement a policy $x \in X$. The policy space X is assumed to be unidimensional, say $X = [0, 1]$.¹⁰ The electorate \mathcal{N} consists of a continuum of citizens. Without loss of generality, we normalize the mass of citizens to one. Each citizen is characterized by an ideal policy $x_n \in X$. Citizens' ideal policies are distributed over X according to some cumulative distribution function F . We make the standard assumptions that F is continuous and strictly increasing over X . Let us denote by m the ideal policy of the median citizen; formally, $F(m) = 1/2$. Without loss of generality, we let $m = 1/2$, and refer to m as the position of the median citizen. A citizen n with ideal policy x_n gets utility $u^n(x) = u(x - x_n)$ from policy $x \in X$. For expositional purposes, we assume that $u^n(x) = u(|x - x_n|)$, i.e., citizens' preferences are symmetric around their ideal policy. Furthermore, we assume that $u(\cdot)$ is a concave and strictly decreasing function, and normalize $u(0)$ to zero.

There is a finite set of potential candidates \mathcal{P} who decide whether to stand for election. In line with the citizen-candidate approach, the set of potential candidates \mathcal{P} is a subset of the electorate \mathcal{N} .¹¹ Being a citizen himself, each potential candidate $i \in \mathcal{P}$ has ideal policy $x_i \in X$ and obtains utility $u^i(x) = u(|x - x_i|)$ from policy x . As a central tenet of the citizen-candidate approach, a potential candidate i cannot credibly commit to implementing any policy other than x_i . In consequence, we shall refer to the ideal policy of a potential candidate as his position. In addition to the utility $u^i(x)$ over the policy outcome, a potential candidate i obtains a utility $\beta \geq 0$ from being elected the policymaker.

¹⁰We assume a unidimensional policy space to facilitate comparison with related contributions (e.g., Cox 1987, 1990, Myerson and Weber 1993, Dellis and Oak 2014), which all assume a unidimensional policy space.

¹¹The assumptions of a finite set of potential candidates and of a continuum of citizens are made to be consistent with the assumption that potential candidates are strategic when making their candidacy decision and sincere when making their voting decision.

In our baseline model, we assume that potential candidates' positions are of three types: a left position $x_L = m - y$, a moderate position $x_M = m$, and a right position $x_R = m + y$, where $y \in (0, 1/2]$ measures the polarization of x_L and of x_R . There are $p_j \in \mathbb{N}$ potential candidates at position $j \in \{L, M, R\}$. Also, our baseline model considers the polar case where $\beta = 0$, i.e., where candidates are purely policy-motivated. We discuss in Section 5 the robustness of our conclusions to the introduction of office-motivation, i.e., $\beta > 0$. In that section we also discuss the generality and limitations of assuming three positions for potential candidates.

3.2. Policymaking process

The policymaking process is modeled as a three-stage game. At the first stage, potential candidates decide, simultaneously and independently, whether to stand for election or not. In standing for election a candidate incurs a utility cost $\delta > 0$. At the second stage, provided there is a non-empty set of candidates, an election is held. Then, at the third stage, the elected candidate chooses and implements a policy. If no candidate stands for election, a default policy $x_0 \in X$ is implemented and the game ends. Following Osborne and Slivinski (1996), we let $u^n(x_0) = -\infty$ for every citizen n . As shall become clear in the next section, this assumption is without loss of generality for our main result (Proposition 1). We now describe each stage in greater detail, proceeding backward.

Policy selection stage

Given that this is the last stage of the game and that candidates cannot credibly commit to the policy they will implement if elected, the policymaker chooses to implement his ideal policy.

Election stage

Let $\mathcal{C} \subseteq \mathcal{P}$ be a non-empty set of candidates who are running for election. We denote the number of candidates by $c \equiv \#\mathcal{C}$, and relabel the candidates from 1 to c in such a way that $x_1 \leq x_2 \leq \dots \leq x_c$.

Let $\alpha^n(\mathcal{C}) = (\alpha_1^n, \dots, \alpha_c^n)$ denote citizen n 's voting decision, where $\alpha^n : \mathcal{C} \rightarrow \{1, \dots, c\}$ is a bijection. For each candidate $i \in \mathcal{C}$, α_i^n indicates the candidate's position on citizen n 's ballot. For example, $\alpha_i^n = 1$ indicates that citizen n ranks

candidate i first, and $\alpha_i^n = c$ indicates that citizen n ranks candidate i last.¹² We denote the vote profile by $\alpha(\mathcal{C})$.

Voting is assumed to be sincere, i.e., each citizen ranks her most preferred candidate first, the next most preferred candidate second, and so on, up to the least preferred candidate, who is ranked last. Formally,

DEFINITION 1 (**Sincere Voting**). A voting decision for citizen n , $\alpha^n(\mathcal{C})$, is sincere if for each pair of candidates, i and j , we have

$$u^n(x_i) > u^n(x_j) \Rightarrow \alpha_i^n < \alpha_j^n.$$

A vote profile $\alpha(\mathcal{C})$ is sincere if every citizen n 's voting decision $\alpha^n(\mathcal{C})$ is sincere. Indifference ties are broken randomly.¹³

We consider two cases, one case in which the election is held under the Plurality rule and another case in which the election is held under the Alternative Vote rule.

Under the Plurality rule, only the first positions on the individual ballots count. More specifically, the vote total of a candidate i is given by

$$V_i(\mathcal{C}, \alpha) = \mu(\{n \in \mathcal{N} : \alpha_i^n = 1\}),$$

where $\mu(S)$ denotes the measure of a set S . The candidate who receives the largest mass of first positions is elected. Ties are broken randomly. The winning set is given by $W(\mathcal{C}, \alpha) \equiv \left\{ i \in \arg \max_{j \in \mathcal{C}} V_j(\mathcal{C}, \alpha) \right\}$. Given the random tie-breaking rule, the probability that a candidate i is elected is given by

$$\pi_i(\mathcal{C}, \alpha) = \begin{cases} \frac{1}{\#W(\mathcal{C}, \alpha)} & \text{if } i \in W(\mathcal{C}, \alpha) \\ 0 & \text{if } i \notin W(\mathcal{C}, \alpha). \end{cases}$$

Under the Alternative Vote rule, all positions on individual ballots may count. More specifically, under the Alternative Vote rule, candidates are eliminated sequentially, one at a time. At the first elimination round, the candidate who receives the smallest mass of first positions is eliminated, and his name is removed

¹²Observe that we implicitly rule out abstention. We do so because of costless voting and complete information; with a finite (possibly large) electorate, vote abstention would be a weakly dominated strategy.

¹³This means that if every citizen in a set S is indifferent between ranking candidate i and candidate j at the k^{th} and $(k+1)^{st}$ positions, then half of them rank candidate i at the k^{th} position and candidate j at the $(k+1)^{st}$ position, while the other half rank candidate j at the k^{th} position and candidate i at the $(k+1)^{st}$ position.

from individual ballots. We then move to the second round, where the candidate who is now ranked the highest on the smallest mass of ballots is eliminated, and his name is removed from individual ballots as well. The process is repeated until only one candidate is left. This candidate is declared the election winner. At each elimination round, ties are broken randomly.

We start by formalizing the first elimination round under the Alternative Vote rule. Given vote profile $\alpha(\mathcal{C})$, let $\mathcal{L}^1(\mathcal{C}, \alpha) \equiv \left\{ i \in \arg \min_{j \in \mathcal{C}} V_j(\mathcal{C}, \alpha) \right\}$ be the set of candidates who are receiving the smallest mass of first positions and, therefore, who are tying for first elimination. The probability that candidate $i \in \mathcal{L}^1(\mathcal{C}, \alpha)$ is the first candidate to be eliminated is equal to $1/\#\mathcal{L}^1(\mathcal{C}, \alpha)$. We denote by $L^1 \in \mathcal{L}^1(\mathcal{C}, \alpha)$ the candidate who is effectively eliminated first.

We proceed recursively for the other elimination rounds. Denote $\mathcal{C}^t = \mathcal{C}^{t-1} \setminus \{L^{t-1}\}$ the set of candidates who have not yet been eliminated at round $t = 2, \dots, c-1$, where $\mathcal{C}^1 \equiv \mathcal{C}$. The mass of citizens who rank candidate $i \in \mathcal{C}^t$ highest among the candidates in \mathcal{C}^t at elimination round t is given by

$$V_i^t(\mathcal{C}^t, \alpha) \equiv \mu \left(\left\{ n \in \mathcal{N} : i = \arg \min_{j \in \mathcal{C}^t} \alpha_j^n \right\} \right).$$

Following the definitions of $\mathcal{L}^1(\mathcal{C}, \alpha)$ and L^1 , we define the set of candidates who are tying for elimination at round t as $\mathcal{L}^t(\mathcal{C}^t, \alpha) \equiv \left\{ i \in \arg \min_{j \in \mathcal{C}^t} V_j^t(\mathcal{C}^t, \alpha) \right\}$, and denote by $L^t \in \mathcal{L}^t(\mathcal{C}^t, \alpha)$ the candidate who is effectively eliminated at round t . We write an elimination sequence $L = (L^1, \dots, L^{c-1})$ and the set of elimination sequences $\Lambda(\mathcal{C}, \alpha)$. The set $\Lambda(\mathcal{C}, \alpha)$ is a singleton if and only if there is never a tie for elimination at any round along the elimination sequence. We write $W_L \equiv \mathcal{C} \setminus \{L^1, \dots, L^{c-1}\}$ the election winner associated with an elimination sequence L . The winning set is given by $W(\mathcal{C}, \alpha) \equiv \{i \in \mathcal{C} : W_L = \{i\} \text{ for some } L \in \Lambda(\mathcal{C}, \alpha)\}$. The probability that a candidate i is elected is given by $\pi_i(\mathcal{C}, \alpha) \equiv \frac{\#\{L \in \Lambda(\mathcal{C}, \alpha) : W_L = \{i\}\}}{\#\Lambda(\mathcal{C}, \alpha)}$.

Candidacy stage

Let $e_i \in \{0, 1\}$ denote the candidacy decision of a potential candidate i , where $e_i = 0$ indicates potential candidate i chooses to not run for election and $e_i = 1$ indicates he chooses to run. We denote the candidacy profile by $e = (e_i)_{i \in \mathcal{P}}$ and the associated set of candidates by $\mathcal{C}(e) \equiv \{i \in \mathcal{P} : e_i = 1\}$. We sometimes write $e = (e_i, e_{-i})$,

where e_{-i} corresponds to the candidacy profile of all potential candidates other than i .

Given a candidacy profile e and a vote profile $\alpha(\cdot)$, the expected utility of a potential candidate i is given by

$$U_i(e, \alpha) = \sum_{j \in \mathcal{P} \cup \{0\}} \pi_j(\mathcal{C}(e), \alpha(\mathcal{C}(e))) u^i(x_j) + \pi_i(\mathcal{C}(e), \alpha(\mathcal{C}(e))) \beta - e_i \delta,$$

where

$$\pi_0(\mathcal{C}(e), \alpha(\mathcal{C}(e))) = \begin{cases} 1 & \text{if } \mathcal{C}(e) = \emptyset \\ 0 & \text{if } \mathcal{C}(e) \neq \emptyset \end{cases}$$

denotes the probability that neither potential candidate stands for election and, therefore, that the default policy x_0 is implemented.

A candidacy profile e^* constitutes a candidacy equilibrium given voter profile $\alpha(\cdot)$ if

$$e_i^* \in \arg \max_{e_i \in \{0,1\}} U_i(e_i, e_{-i}^*; \alpha)$$

for every potential candidate $i \in \mathcal{P}$.

Political equilibrium

An equilibrium is a pair $(e^*, \alpha^*(\cdot))$ where: (i) $\alpha^*(\mathcal{C})$ is a sincere vote profile for every non-empty set of candidates \mathcal{C} ; and (ii) e^* is a candidacy profile given vote profile $\alpha^*(\cdot)$. It is easy to show by construction that an equilibrium exists for any configuration of positions (x_L, x_M, x_R) .

3.3. Defining policy polarization

Intuitively, we say that a voting rule supports more policy polarization than another voting rule if 1) it supports the adoption of more extreme policies (i.e., policies that lie further away from the median m) and 2) it does not support the adoption of more moderate policies (i.e., policies that lie closer to the median). We say that a voting rule supports the adoption of a policy if for a configuration of positions (x_L, x_M, x_R) that includes this policy, an equilibrium exists where a potential candidate at this position is elected with a strictly positive probability.

Formally, let $Y^r \subseteq (0, 1/2]$ be the set of polarization levels $y \in (0, 1/2]$ for which, given configuration of positions $(x_L, x_M, x_R) = (m - y, m, m + y)$, an equilibrium (e^*, α^*) exists under voting rule r , in which $\pi_i(\mathcal{C}(e^*), \alpha^*(\mathcal{C}(e^*))) > 0$ for some

potential candidate $i \in \mathcal{P}$ with ideal policy $x_i \in \{m - y, m + y\}$. In words, Y^r is the set of polarization levels which can be supported in equilibrium by voting rule r . Observe that the degree of policy polarization supported by a voting rule r is minimal if $Y^r = \emptyset$, i.e., if the position of the median citizen (here Condorcet winner) is adopted with probability one in every equilibrium and for every configuration of positions.

We are now ready to define formally our notion of policy polarization.

DEFINITION 2 (Policy polarization). A voting rule r supports more policy polarization than a voting rule s if and only if $Y^r \neq Y^s$, $Y^r \neq \emptyset$ and the following two conditions hold:

- (i) for each $y \in Y^r \setminus Y^s$ and all $z \in Y^s$, $y > z$; and
- (ii) for each $y \in Y^s \setminus Y^r$ and all $z \in Y^r$, $y < z$.

In words, a voting rule r supports more policy polarization than a voting rule s when 1) every policy which adoption can be supported by r but not by s is more extreme than any of the policies which adoption can be supported by s , and 2) every policy which adoption can be supported by s but not by r is more moderate than any of the policies which adoption can be supported by r .

4. ANALYSIS

In this section, we start by providing a complete characterization of equilibria under the Plurality rule and under the Alternative Vote rule. We then compare the degree of policy polarization supported by the two rules.

4.1. Equilibrium characterization

We start by establishing some results on the equilibrium number and locations of candidates under the two rules.

LEMMA 1. *Let (e^*, α^*) be any equilibrium under the Plurality rule or under the Alternative Vote rule. We have $x_i \neq x_j$ for all candidates $i, j \in \mathcal{C}(e^*)$, $i \neq j$.*

Thus, no two candidates are standing at the same position, whether the election is held under the Plurality rule or under the Alternative Vote rule.

To understand the intuition underlying this result, consider a candidacy profile in which at least two candidates are running at the same position. Clearly, this

cannot be an equilibrium candidacy profile if all candidates are at the same position; one of these candidates would be better off not running since by doing so he would save the candidacy cost while the policy outcome would be unchanged. Now consider a candidacy profile with several positions at which candidates are running. The intuition as to why this profile cannot be an equilibrium differs under the Plurality rule and under the Alternative Vote rule. Under the Plurality rule, the result relies on the splitting-the-vote effect. Specifically, two or more candidates at the same position would split their votes, thereby helping the election of a candidate standing at another position. By contrast, under the Alternative Vote rule, the result relies on the sequential elimination process. Specifically, the presence of two or more candidates at the same position would lengthen the elimination sequence without changing the probability distribution over policy outcomes since the votes of an eliminated candidate are transferred to his closest neighbor(s). It follows that under both rules all the candidates in excess of one at a position would be strictly better off deviating by not running since they would each save on the candidacy cost while (weakly) increasing the probability with which their ideal policy is adopted. This would contradict the premise that the candidacy profile is part of an equilibrium.

We now proceed to characterize the sets of equilibria under the two rules. For this purpose, we partition the set of equilibria into three subsets, namely, the subsets of 1-, 2- and 3-position equilibria, in which there are candidates at one, two and three positions, respectively.

We start by characterizing the 1-position equilibria.

LEMMA 2. *In any 1-position equilibrium under the Plurality rule or under the Alternative Vote rule, a single candidate runs unopposed. An equilibrium in which potential candidate $i \in \mathcal{P}$ runs unopposed exists under either voting rule if and only if*

1. $x_i = x_M$, or
2. $x_i \in \{x_L, x_R\}$ and $\delta > -\frac{u(x_L - x_R)}{2}$.

The intuition underlying this result runs as follows. First, we already know from Lemma 1 that in equilibrium no two candidates are running at the same position. It follows trivially that in any 1-position equilibrium, only one candidate is running for office. Second, the fact that only one candidate runs for office implies that the 1-

position equilibria are equivalent under the Plurality rule and the Alternative Vote rule.¹⁴ Finally, it must be that no other potential candidate (at another position) is willing to enter the race. This happens if: (1) candidate i 's position is x_M , since he would be preferred by a majority of citizens and the entrant would be defeated with probability one; or (2) candidate i 's position is x_L or x_R , and the candidacy cost exceeds the utility gain for a candidate at the other position in $\{x_L, x_R\}$ entering the race and tying for first place.¹⁵

Next, we proceed to characterize the 2-position equilibria. We define $\underline{x} \equiv \frac{x_L + x_M}{2}$ as the ideal policy of a citizen who is indifferent between x_L and x_M . Likewise, we define $\bar{x} \equiv \frac{x_M + x_R}{2}$ as the ideal policy of a citizen who is indifferent between x_M and x_R . The following lemma provides a complete characterization of the 2-position equilibria under each of the two voting rules.

LEMMA 3. *A 2-position equilibrium (e^*, α^*) exists under the Plurality rule or under the Alternative Vote rule if and only if the following three conditions hold:*

(i) *The candidacy profile e^* is such that $\mathcal{C}(e^*) = \{i, j\}$, with $x_i = x_L$ and $x_j = x_R$;*

(ii) $\delta \leq -\frac{u(|x_L - x_R|)}{2}$; *and*

(iii-Plurality rule) *Under the Plurality rule, one of the following holds true*

1. $F(\bar{x}) - F(\underline{x}) > \max\{F(\underline{x}), 1 - F(\bar{x})\}$ *and* $\delta > -u(|x_M - x_L|)$.
2. $F(\bar{x}) - F(\underline{x}) = \max\{F(\underline{x}), 1 - F(\bar{x})\} > \min\{F(\underline{x}), 1 - F(\bar{x})\}$ *and* $\delta > -\frac{u(|x_M - x_L|)}{2}$.
3. $F(\bar{x}) - F(\underline{x}) = \max\{F(\underline{x}), 1 - F(\bar{x})\} = \min\{F(\underline{x}), 1 - F(\bar{x})\}$ *and* $\delta > -\frac{u(|x_M - x_L|)}{3}$.
4. $F(\bar{x}) - F(\underline{x}) < \max\{F(\underline{x}), 1 - F(\bar{x})\}$.

(iii-Alternative Vote rule) *Under the Alternative Vote rule, one of the following holds true*

1. $F(\bar{x}) - F(\underline{x}) > \min\{F(\underline{x}), 1 - F(\bar{x})\}$ *and* $\delta > -u(|x_M - x_L|)$.
2. $F(\bar{x}) - F(\underline{x}) = \min\{F(\underline{x}), 1 - F(\bar{x})\} < \max\{F(\underline{x}), 1 - F(\bar{x})\}$ *and* $\delta > -\frac{u(|x_M - x_L|)}{2}$.

¹⁴Clearly, both voting rules elect the single candidate. If the candidate deviates by not running, then the default policy x_0 is implemented under both rules. If another potential candidate deviates by entering the race, then there will be two candidates running and the candidate preferred by the median citizen is elected under both voting rules.

¹⁵Given the (weak) concavity of the utility function $u(\cdot)$, the latter condition is sufficient to deter a potential candidate to enter the race against a candidate at x_L or x_R .

3. $F(\bar{x}) - F(\underline{x}) = \min\{F(\underline{x}), 1 - F(\bar{x})\} = \max\{F(\underline{x}), 1 - F(\bar{x})\}$ and $\delta > -\frac{2}{3}u(|x_M - x_L|)$.
4. $F(\bar{x}) - F(\underline{x}) < \min\{F(\underline{x}), 1 - F(\bar{x})\}$.

Condition (i) specifies that in any 2-position equilibrium, candidates are running at x_L and x_R , whether the election is held under the Plurality rule or under the Alternative Vote rule. The intuition underlying this condition is related to the fact that there are only two candidates running for election (by Lemma 1). It follows that if one candidate were positioned at x_M , he would be strictly preferred to the other candidate by a majority of citizens and would be elected outright. The other candidate would therefore be strictly better off deviating by not running since he would save on the candidacy cost while the election outcome would be unchanged. This would contradict the premise that the candidacy profile is part of an equilibrium.

Condition (ii) specifies a lower bound on the degree of polarization between x_L and x_R . Essentially, the two positions must be sufficiently polarized that neither candidate would be better off deviating by not running. Specifically, the electorate is equally split between the two candidates, and each candidate is elected with probability 1/2. If one candidate were to deviate by not running, he would save on the candidacy cost, but the other candidate would now be elected outright. The benefit from the deviation is then equal to the candidacy cost δ , while the cost is equal to $\frac{u(|x_L - x_R|)}{2} - u(|x_L - x_R|) = -\frac{u(|x_L - x_R|)}{2}$. For a candidate to not be willing to deviate, it must then be that the cost $-\frac{u(|x_L - x_R|)}{2}$ is at least as large as the benefit δ . This condition is the same under the Plurality rule as under the Alternative Vote rule given that the deviation cost is associated with sets of one and two candidates and that the two voting rules differ only with three or more candidates running.

Finally, condition (iii) specifies an upper bound on the degree of polarization between x_L and x_R . Essentially, the two positions must not be so polarized that a potential candidate at x_M would want to enter the race. Contrary to the lower bound on policy polarization that was specified in condition (ii), the upper bound specified in condition (iii) varies with the voting rule (since it involves sets with three candidates). Specifically, if a potential candidate at x_M were to enter the race, then: 1) all citizens with ideal policy to the left of \underline{x} , preferring x_L to x_M and

x_R , would rank the candidate at x_L first; 2) all citizens with ideal policy between \underline{x} and \bar{x} , preferring x_M to x_L and x_R , would rank the candidate at x_M first; and 3) all citizens with ideal policy to the right of \bar{x} , preferring x_R to x_M and x_L , would rank the candidate at x_R first. First-place vote totals would then be equal to $F(\underline{x})$, $F(\bar{x}) - F(\underline{x})$ and $1 - F(\bar{x})$ for the candidate at x_L , the candidate at x_M and the candidate at x_R , respectively. For a potential candidate at x_M to be willing to enter the race, he must anticipate that he will be elected with positive probability. Under the Plurality rule, a candidate at x_M is elected with positive probability if and only if he receives a plurality of votes, i.e., $F(\bar{x}) - F(\underline{x}) \geq \max\{F(\underline{x}), 1 - F(\bar{x})\}$. Under the Alternative Vote rule, a candidate at x_M is elected with positive probability if and only if he does not receive strictly fewer first-place votes than any other candidate, i.e., $F(\bar{x}) - F(\underline{x}) \geq \min\{F(\underline{x}), 1 - F(\bar{x})\}$. This is because the candidate at x_M is the Condorcet winner and would therefore defeat any other candidate in a pairwise contest. Under the Alternative Vote rule, the candidate at x_M thus necessarily wins the election as long as he is not eliminated first. As we shall see in the next subsection, this difference between the Plurality rule and the Alternative Vote rule is key for the degree of policy polarization that each of these two voting rules can support. Subcondition (iii.1) considers the case where a candidate at x_M would win outright, in which case a potential candidate at x_M does not want to enter if and only if the candidacy cost δ exceeds the utility gain $-u(|x_M - x_L|)$ of getting x_M implemented with probability one instead of getting x_L and x_R implemented with probability 1/2 each. Subconditions (iii.2) and (iii.3) consider the cases where a candidate at x_M would tie for first place (resp. first elimination) under the Plurality rule (resp. under the Alternative Vote rule) with one or two of the left and right candidates. Finally, subcondition (iii.4) considers the case where a candidate at x_M would be elected with probability zero and, therefore, would not want to enter the race.

Our last lemma rules out the existence of 3-position equilibria under either of the two voting rules.

LEMMA 4. *There is no 3-position equilibrium, whether the election is held under the Plurality rule or under the Alternative Vote rule.*

The intuition underlying this result is the same whether the election is held under the Plurality rule or under the Alternative Vote rule. Essentially, the result

follows from the fact that if a 3-position equilibrium were to exist, there would be only one candidate standing for election at each position (by Lemma 1). Given that candidacy is costly and that the utility function is (weakly) concave, the candidate at x_L or the candidate at x_R (or both) would be better off deviating by not running and letting the candidate at x_M winning outright (which would necessarily happen since the candidate at x_M is the Condorcet winner and there would be only two candidates left).

4.2. Comparing policy polarization

We are now ready to establish our main result.

PROPOSITION 1. *The Alternative Vote rule supports weakly less policy polarization than the Plurality rule.*

To understand this result, we start by noticing that the set of policies which can be supported in equilibrium is an interval centered at the median m .¹⁶ To see this, first recall from Lemma 4 that the set of 3-position equilibria is empty under both voting rules. Hence, for both voting rules the set of policies which can be supported is characterized by the 1- and 2-position equilibria only. Now, recall from Lemma 2 that the set of 1-position equilibria is equivalent under both voting rules. Moreover, together condition 2 of Lemma 2 and condition (ii) of Lemma 3 imply that 1) every 1-position equilibrium is more moderate than any 2-position equilibrium, and 2) the upper-bound on polarization for the 1-position equilibria coincides with the lower-bound on polarization for the 2-position equilibria. Hence the result that the set of policies which can be supported in equilibrium is an interval centered around the median.

This observation implies that the difference in the levels of policy polarization that the Alternative Vote rule and the Plurality rule can support is captured by the upper-bounds on polarization for the 2-position equilibria. These upper-bounds are given by condition (iii) of Lemma 3. More specifically, this condition specifies that positions x_L and x_R cannot be too polarized so that no potential candidate at x_M would be better off deviating from his candidacy strategy by standing for election. How polarized x_L and x_R can be while still deterring a moderate from entering the race depends on the voting rule.

¹⁶This observation is formally established in Lemma 5 in the Appendix.

Under the Plurality rule, a moderate candidate would need to receive a plurality of votes to be elected, i.e., to receive more votes than the candidate at x_L and than the candidate at x_R . Formally, a candidate at x_M entering the race would receive votes from all citizens with ideal policy between \underline{x} and \bar{x} ; his vote share would thus be equal to $F(\bar{x}) - F(\underline{x})$. At the same time, the candidate at x_L (resp. x_R) would receive votes from all citizens with ideal policy to the left of \underline{x} (resp. to the right of \bar{x}), and his vote share would thus be equal to $F(\underline{x})$ (resp. $1 - F(\bar{x})$). Hence, a potential candidate at x_M entering the race would be defeated, and therefore necessarily deterred from entering the race, only if $F(\bar{x}) - F(\underline{x}) < \max\{F(\underline{x}), 1 - F(\bar{x})\}$.

Under the Alternative Vote rule, a moderate candidate would only need to not be eliminated at the first count in order to be elected, i.e., he would only need to receive more first-place votes than the candidate at x_L or than the candidate at x_R . This is because a candidate at x_M would be the Condorcet winner and, at the second count, would defeat any of the other two candidates. Formally, a potential candidate at x_M entering the race would be defeated, and therefore necessarily deterred from entering the race, only if $F(\bar{x}) - F(\underline{x}) < \min\{F(\underline{x}), 1 - F(\bar{x})\}$.

Hence, it is easier to deter a potential candidate at x_M from entering the race under the Plurality rule than under the Alternative Vote rule. Thus, x_L and x_R can be more polarized under the Plurality rule than under the Alternative Vote rule while still deterring a moderate from entering the race. Hence the result in Proposition 1.

Thus, Proposition 1 shows that the argument that the Alternative Vote rule supports less policy polarization than the Plurality rule is robust to endogenizing candidacy. This result contrasts with findings in Dellis and Oak (2014) which shows that for non-ranking scoring rules a similar argument is not robust to endogenous candidacy. A key difference between those rules and the Alternative Vote rule is that they induce multiple candidates clustering, which the Alternative Vote rule deters. More formally, those rules do not satisfy the independence of clones criterion while the Alternative Vote rule does.

Proposition 1 establishes that the Alternative Vote rule supports weakly less policy polarization than the Plurality rule. However, Proposition 1 does not rule out the possibility that the Alternative Vote rule supports as much policy polarization as

the Plurality rule. The following corollary establishes that the two rules generically support as much policy polarization when citizens' ideal policies are distributed symmetrically around the median m . By 'generically' we mean that the sets of polarization levels Y^{AVR} and Y^{PR} under the Alternative Vote rule and the Plurality rule, respectively, have the same infimum and the same supremum (i.e., the two sets may differ only on whether they are open or half-open intervals).

COROLLARY 1. *Suppose $F(x) = 1 - F(1 - x)$ for every $x \in [0, 1]$. Then, the Alternative Vote rule generically supports as much policy polarization as the Plurality rule.*

It follows that the Alternative Vote rule supports less policy polarization than the Plurality rule only if ideal policies are asymmetrically distributed around the median m . Examples where it is the case are easy to construct. Thus, whether the Alternative Vote rule supports strictly less policy polarization than the Plurality rule boils down to the empirical question whether actual distributions of ideal policies are symmetric around the median or not.

5. CONCLUSION AND DISCUSSION

The Alternative Vote rule (including its variants) is currently used in political elections in a variety of contexts, e.g., elections to the Australian Lower House, Irish Presidential elections, mayoral races in many cities around the world. Moreover, its broader adoption for single seat elections has been advocated by several citizen groups (e.g., the Center for Voting and Democracy) and has recently been the object of several referenda around the world (e.g., in the Canadian province of British Columbia in 2009, in the UK in 2011). One of the arguments often put forward to justify the adoption of the Alternative Vote rule is that it would benefit moderates. This paper attempts to formally explore whether the proponents of the Alternative Vote rule have a point once we take strategic candidacy decisions into account.

We find that the Alternative Vote rule does support (weakly) less policy polarization than the Plurality rule. The intuition underlying this result comes from the fact that the Alternative Vote rule weakens the *squeezing* of moderate candidates compared to the Plurality rule: under the Plurality rule a moderate candidate

would get squeezed in between the left and right extremists who capture votes on either sides of him whereas under the Alternative Vote rule the centrist candidate survives the first elimination and then emerges victorious in the second. This means that polarized equilibria, i.e., those with two extremists running against each other, are harder to support under the Alternative Vote rule due to a credible threat of moderate entry.

We also find that, as with the Plurality rule, the Alternative Vote rule deters multiple (policy-motivated) candidates from standing for election at the same position. The intuition however is different under the Plurality rule and the Alternative Vote rule. Under the Plurality rule, the result follows because several candidates with a similar platform would run the risk of splitting their votes, thereby allowing the election of less-preferred candidates. Under the Alternative Vote rule, the result follows because this rule satisfies the Independence of Clones criterion; the presence of several candidates with similar platforms would lengthen the elimination sequence without affecting the policy outcome. Finally, our analysis suggests that, as the Plurality rule, the Alternative Vote rule tends to favor a two-party system (i.e., a situation with candidates at only one or two positions). This result is consistent with empirical observations. It happens in our setting because of strategic candidacy behavior. These results stand in contrast to the equilibria exhibited under other voting rules which have also been widely advocated, in particular Approval Voting. Our previous work (Dellis and Oak, 2014) has shown that Approval Voting is prone to candidate clustering and can support greater policy polarization vis-à-vis the Plurality rule.

In the course of our analysis we made several assumptions. Some of these assumptions were made to simplify the analysis. One of these assumptions was that candidates are purely policy-motivated, i.e., the ego rent $\beta = 0$. This assumption makes for a clean analysis and sharp results. It is easy to see that our qualitative results carry over to cases where $\beta < 2\delta$, where δ stands for the candidacy cost. This follows because for those values of β , candidates are still deterred from entering the race at the same position. For cases where $\beta > 2\delta$, multiple candidates may share the same platform under the Alternative Vote rule, but the set of policies which adoption can be supported in equilibrium is still an interval, although with a lower upper-bound on the degree of policy polarization that can be supported. Finally,

for cases where $\beta > 3\delta$, 3-position equilibria may now exist (contrary to what we found when $\beta = 0$), but the set of policies which adoption can be supported by 3-position equilibria is of measure zero. Hence, allowing for office-motivation reduces generically the degree of policy polarization that can be supported under the Alternative Vote rule.

Another simplifying assumption was the presence of potential candidates at only three positions. This assumption allows us to capture in an easy way the squeezing of moderates due to the presence of the extremists. Under the Plurality rule this assumption—that there are only three potential positions—imposes no restriction. Under the Alternative Vote rule, our qualitative results carry over to settings with four positions and, under additional conditions, to settings with five positions.¹⁷ The difficulty of establishing results for cases with more than three positions arises due to the fact that the Alternative Vote rule violates the monotonicity property (Brams and Fishburn, 1984). This feature of the Alternative Vote rule implies that the presence or absence of a candidate may in some cases harm his direct neighbors and in other cases improve their electoral prospects.

There is also a number of assumptions that were made to facilitate comparison with the related literature and to isolate the effect of endogenizing candidacy. These assumptions are the unidimensionality of the policy space, the completeness of information, and the focus on a one-shot election. Considering dynamic elections, introducing information incompleteness or allowing a multidimensional policy space goes beyond the scope of the present paper, and is left for future work.

REFERENCES

- [1] **Besley, T. and S. Coate.** 1997. “An economic model of representative democracy.” *Quarterly Journal of Economics*, 112: 85-114.
- [2] **Brams, S. and P. Fishburn.** 1984. “Some logical defects of the Single Transferable Vote.” In: *Choosing an Electoral System: Issues and Alternatives*, A. Lijphart and B. Grofman (eds), Praeger: New York.

¹⁷The formal proofs are really tedious and are therefore omitted here. They are available from the authors.

- [3] **Brusco, S. and J. Roy.** 2011. "Aggregate uncertainty in the citizen candidate model yields extremist parties." *Social Choice and Welfare*, 36(1): 83-104.
- [4] **Callander, S.** 2005. "Electoral competition in heterogeneous districts." *Journal of Political Economy*, 113(5): 1116-1145.
- [5] **Cox, G.** 1987. "Electoral equilibrium under alternative voting institutions." *American Journal of Political Science*, 31: 82-108.
- [6] **Cox, G.** 1990. "Centripetal and centrifugal incentives in electoral systems." *American Journal of Political Science*, 34: 903-935.
- [7] **Dellis, A.** 2009. "Would letting people vote for multiple candidates yield policy moderation?" *Journal of Economic Theory*, 144: 772-801.
- [8] **Dellis, A. and M. Oak.** 2006. "Approval Voting with endogenous candidates." *Games and Economic Behavior*, 54: 47-76.
- [9] **Dellis, A. and M. Oak.** 2014. "Multiple Votes, Multiple Candidacies and Polarization." *University of Adelaide Working Paper*.
- [10] **Dutta, B., M. Jackson and M. Le Breton.** 2001. "Strategic candidacy and voting procedures." *Econometrica*, 69: 1013-1037.
- [11] **Duverger, M.** 1954. "*Political Parties: Their Organization and Activity in the Modern State.*" Wiley, New York.
- [12] **Eguia, J.** 2007. "Citizen candidates under uncertainty." *Social Choice and Welfare*, 29(2): 317-331.
- [13] **Farrell, D. and I. McAllister.** 2006. "*The Australian Electoral System. Origins, Variations and Consequences.*" University of New South Wales Press: Sydney, Australia.
- [14] **Feddersen, T., I. Sened and S. Wright.** 1990. "Rational voting and candidate entry under Plurality rule." *American Journal of Political Science*, 34(4): 1005-1016.
- [15] **Forsythe, R., T. Rietz, R. Myerson and R. Weber.** 1996. "An experimental study of voting rules and polls in three-candidate elections." *International Journal of Game Theory*, 25(3): 355-383.

- [16] **Grofman, B. and S. Feld.** 2004. “If you like the alternative vote (a.k.a. the instant runoff), then you ought to know about the Coombs rule.” *Electoral Studies*, 23: 641-659.
- [17] **Grosser J. and T. Palfrey.** 2014. “Candidate entry and political polarization: An antimedial voter theorem.” *American Journal of Political Science*, 58(1): 127–143.
- [18] **Jesse, N.** 2000. “A sophisticated voter model of preferential electoral systems.” In: *Elections in Australia, Ireland, and Malta under the Single Transferable Vote*, S. Bowler and B. Grofman (eds), Michigan University Press.
- [19] **Lijphart, A.** 1994. “*Electoral Systems and Party Systems.*” Oxford University Press: Oxford.
- [20] **Matakos, K., O. Troumpounis and D. Xeftaris.** 2014. “Electoral Rule Disproportionality and Platform Polarization.” *Mimeo.*
- [21] **Merrill, S.** 1988. “*Making Multicandidate Elections More Democratic.*” Princeton University Press: Princeton, NJ.
- [22] **Morelli, M.** 2004. “Party formation and policy outcomes under different electoral systems.” *Review of Economic Studies*, 71: 829-853.
- [23] **Myerson, R. and R. Weber.** 1993. “A theory of voting equilibria.” *American Political Science Review*, 87: 102-114.
- [24] **Osborne, M.** 1993. “Candidate positioning and entry in a political competition.” *Games and Economic Behavior*, 5: 133–151.
- [25] **Osborne, M.** 2000. “Entry-detering policy differentiation by electoral candidates.” *Mathematical Social Sciences*, 40: 41–62.
- [26] **Osborne, M. and A. Slivinski.** 1996. “A model of political competition with citizen-candidates.” *Quarterly Journal of Economics*, 111: 65-96.
- [27] **Palfrey, T.** 1984. “Spatial equilibrium with entry.” *Review of Economic Studies*, 51: 139-156.
- [28] **Riker, W.** 1982. “The two-party system and Duverger’s law: An essay on the history of political science.” *American Political Science Review*, 76: 753–766.

- [29] **Tideman, N.** 1987. “Independence of clones as a criterion for voting rules.” *Social Choice and Welfare*, 4: 185-206.
- [30] **van der Straeten, K., J-F. Laslier, N. Sauger and A. Blais.** 2010. “Strategic, sincere, and heuristic voting under four election rules: An experimental study.” *Social Choice and Welfare*, 35: 435-472.
- [31] **Weber, S.** 1992. “On hierarchical spatial competition.” *Review of Economic Studies*, 59: 407-425.
- [32] **Weber, S.** 1998. “Entry deterrence in electoral spatial competition.” *Social Choice and Welfare*, 15: 31-56.
- [33] **Xeferis, D.** 2014. “Mixed equilibria in runoff elections.” *Games and Economic Behavior*, 87: 619-623.

APPENDIX

Proof of Lemma 1. We prove the result by contradiction. Let (e, α) be an equilibrium, with the corresponding set of candidates $\mathcal{C}(e)$. Assume by way of contradiction that $x_i = x_j$ for some $i, j \in \mathcal{C}(e)$, $i \neq j$.

First, we consider the case where the election is held under the Plurality rule. Key to observe is that several candidates at a position split their votes. To see this, we start by introducing some notation. Let $X(e) \equiv \{x \in \{x_L, x_M, x_R\} : x_k = x \text{ for some } k \in \mathcal{C}(e)\}$ be the set of positions at which candidates are standing. Also, let $c_i \equiv \#\{k \in \mathcal{C}(e) : x_k = x_i\}$ be the number of candidates (including candidate i) standing at the same position as candidate i . The vote total of any candidate j with $x_j = x_i$ is given by

$$V_j(\mathcal{C}(e), \alpha) = \frac{\mu \left(\left\{ n \in \mathcal{N} : x_i \in \arg \max_{x \in X(e)} u^n(x) \right\} \right)}{c_i}.$$

Suppose candidate i were to deviate by not running for election. We denote by \tilde{e} the candidacy profile after the deviation where $\tilde{e}_i = 0$ and $\tilde{e}_k = e_k$ for all $k \in \mathcal{P} \setminus \{i\}$. The set of candidates is now $\mathcal{C}(\tilde{e}) = \mathcal{C}(e) \setminus \{i\}$ and candidates' vote totals are

$$\begin{cases} V_j(\mathcal{C}(\tilde{e}), \alpha) = \frac{c_i}{c_i-1} V_j(\mathcal{C}(e), \alpha) > V_j(\mathcal{C}(e), \alpha) & \text{for all } j \in \mathcal{C}(\tilde{e}), x_j = x_i \\ V_k(\mathcal{C}(\tilde{e}), \alpha) = V_k(\mathcal{C}(e), \alpha) & \text{for all } k \in \mathcal{C}(\tilde{e}), x_k \neq x_i. \end{cases}$$

Hence the deviation weakly increases the probability that x_i is implemented. It follows that candidate i 's expected utility is such that $U_i(\tilde{e}, \alpha) \geq U_i(e, \alpha) + \delta > U_i(e, \alpha)$. Hence candidate i is strictly better off deviating, which contradicts that e is an equilibrium candidacy profile.

Second, we consider the case where the election is held under the Alternative Vote rule. The key here is that policy outcomes depend on the support for each position, not on whether there are multiple candidates or a single candidate at a position.

Let \tilde{e} be the candidacy profile in which $\tilde{e}_i = 0$ and $\tilde{e}_k = e_k$ for all $k \in \mathcal{P} \setminus \{i\}$. Also, let $x(e)$ and $x(\tilde{e})$ denote the probability distributions over $\{x_L, x_M, x_R\}$ associated with candidacy profiles e and \tilde{e} , respectively.

If e is a 1-position candidacy profile (i.e., $x_h = x_k$ for all $h, k \in \mathcal{C}(e)$), then we trivially get $x(\tilde{e}) = x(e)$. Hence $U_i(\tilde{e}, \alpha) > U_i(e, \alpha)$ given candidacy cost $\delta > 0$, which contradicts that e is an equilibrium candidacy profile.

Suppose now that e is a 2-position candidacy profile (i.e., there are two positions at which candidates are standing). We first observe that x_M cannot be one of the two positions. To see this, assume the contrary. Pick an elimination sequence $L \in \Lambda(\mathcal{C}(e), \alpha)$. One possibility is that $x_k = x_M$ for the two candidates in \mathcal{C}^{c-1} , i.e., the two candidates who have not yet been eliminated at round $c - 1$. The other possibility is that there is a round $t' < c - 1$ at which there is only one candidate h left at x_M . Candidate h is then the Condorcet winner and his vote total $V_h^t(\mathcal{C}^t, \alpha) > 1/2$ at every round $t = t', \dots, c - 1$. Thus, for both possibilities a candidate at x_M is elected. Since this is true for any elimination sequence, the candidate(s) standing at the other position would be strictly better off deviating by not running, a contradiction. Hence candidates must be standing at x_L and x_R .

Observe that the electorate is equally divided between x_L and x_R . It follows that in any elimination sequence $L \in \Lambda(\mathcal{C}, \alpha)$ for $\mathcal{C} = \mathcal{C}(e), \mathcal{C}(\tilde{e})$, the candidates h and k who have not yet been eliminated at round $c - 1$ are such that: (i) $x_h = x_L$ and $x_k = x_R$; and (ii) $V_h^{c-1}(\mathcal{C}^{c-1}, \alpha) = V_k^{c-1}(\mathcal{C}^{c-1}, \alpha) = 1/2$. Hence $x(\tilde{e}) = x(e)$, with x_L and x_R adopted with probability $1/2$ each. It follows that $U_i(\tilde{e}, \alpha) = U_i(e, \alpha) + \delta > U_i(e, \alpha)$, a contradiction.

We get from above that e must be a 3-position candidacy profile (i.e., there must

be candidates standing at each of the three positions). W.l.o.g. we transform elimination sequences such that candidate i is the first candidate at x_i to be eliminated. Formally, for every $L \in \Lambda(\mathcal{C}(e), \alpha)$, let $\underline{t} = \min\{t \in \{1, \dots, c-1\} : x_{L^t} = x_i\}$. If $L^{\underline{t}} = \{i\}$, then we keep the elimination sequence unchanged. If $L^{\underline{t}} \neq \{i\}$, then we swap candidates i and $L^{\underline{t}}$ in the elimination sequence. By an abuse of notation, we write $\Lambda(\mathcal{C}(e), \alpha)$ the set of transformed elimination sequences.

W.l.o.g. suppose that at the earliest round at which candidate i is eliminated, there are still candidates standing at each of the three positions (otherwise the arguments above apply straightforwardly). There are two possibilities to consider.

1. There exists a round $t \in \{1, \dots, c-1\}$ such that $L^t = \{i\}$ for every $L \in \Lambda(\mathcal{C}(e), \alpha)$, i.e., at round t the candidates at x_i are not in a tie for elimination with candidates at other positions. We partition $\Lambda(\mathcal{C}(e), \alpha)$ into families which each contains all the elimination sequences with a same elimination ordering of the candidates at positions other than x_i . Observe that the adopted policy is the same in every elimination sequence belonging to a family λ . Moreover, each family λ contains $c_i!$ elimination sequences.

If candidate i were to deviate by not running, there would be $(c_i - 1)!$ elimination sequences in every family λ . Moreover, each elimination sequence \tilde{L} in a family λ of $\Lambda(\mathcal{C}(\tilde{e}), \alpha)$ corresponds to an elimination sequence L in the family λ of $\Lambda(\mathcal{C}(e), \alpha)$ such that

$$\begin{cases} \tilde{L}^t = L^t & \text{for } t = 1, \dots, \underline{t} - 1 \\ \tilde{L}^t = L^{t+1} & \text{for } t = \underline{t}, \dots, c - 2. \end{cases}$$

Hence the deviation of candidate i would reduce in each family λ the number of sequences by the same number c_i and would shorten every elimination sequence by one round, but $x(\tilde{e}) = x(e)$. It follows that $U_i(\tilde{e}, \alpha) > U_i(e, \alpha)$, a contradiction.

2. There exists a pair of elimination sequences $L, \tilde{L} \in \Lambda(\mathcal{C}(e), \alpha)$ such that $L^t = \{i\}$ and $\tilde{L}^t \neq \{i\}$ for some $t \in \{1, 2, \dots, c-1\}$, i.e., at the earliest round at which candidate i is eliminated, \underline{t} , the candidates at x_i are in a tie for elimination with candidate(s) at other position(s). If at round \underline{t} neither of the candidates in the tie is standing alone at his position, then the same argument as above applies since, at this elimination stage, votes are transferred to other candidates at the same position, implying that the presence of candidate i in

the race does not matter for the order in which positions are eliminated. If instead at round \underline{t} one candidate involved in the tie is standing alone at his position, then the presence of candidate i in the race can matter for the order in which positions are eliminated since the votes of an eliminated position are transferred to candidate(s) at other position(s). The latter is true only for families of elimination sequences in which candidate i is eliminated after a candidate standing alone at his position is eliminated. For each family in which the first candidate (involved in the tie) standing alone at his position to be eliminated is at x_L or x_R , then his votes are transferred to the candidate(s) at x_M , and x_M is adopted in this family and in the corresponding family of $\Lambda(\mathcal{C}(\tilde{e}), \alpha)$. For each family in which the candidate is instead at x_M , then his votes are transferred to x_L and x_R , and each of these two positions is adopted with probability $1/2$ in this family and in the corresponding family of $\Lambda(\mathcal{C}(\tilde{e}), \alpha)$. To sum up, we get $x(\tilde{e}) = x(e)$ again, and thus the contradiction.

■

Proof of Lemma 2. Let (e, α) be a 1-position equilibrium under either the Plurality rule or the Alternative Vote rule. We already know from Lemma 1 that there is a single candidate, say candidate i , running for election. The remaining of the proof is identical to the proof of Lemma 1 in Dellis and Oak (2014). ■

Proof of Lemma 3. Let (e, α) be a 2-position equilibrium under either the Plurality rule or the Alternative Vote rule.

First, we establish the necessity of condition (i). We already know from Lemma 1 that there must be only two candidates running for election, one at each position. Without loss of generality we call the two candidates i and j , and let $x_i < x_j$. That we must have $x_i = x_L$ and $x_j = x_R$ follows because the candidate at x_M would be the Condorcet winner and, since there are only two candidates, would be elected with probability one. The other candidate would then be better off not running since he would then save on the candidacy cost while the election outcome would be left unchanged.

Second, we establish the necessity of condition (ii). For (e, α) to be an equilibrium, it must be that neither of the candidates would be better off deviating by not

running. Given condition (i), we have that each of the two candidates is elected with probability $1/2$. Candidate i 's expected utility is thus equal to $\frac{u(|x_L - x_R|)}{2} - \delta$. If candidate i were to deviate by not running, then candidate j would be elected outright and candidate i 's expected utility would be equal to $u(|x_L - x_R|)$. Thus, candidate i does not want to deviate only if

$$\frac{u(|x_L - x_R|)}{2} - \delta \geq u(|x_L - x_R|).$$

Hence the necessity of condition (ii).

Third, we establish the necessity of condition (iii). Since (e, α) is an equilibrium, it must be that neither potential candidate at x_M would be better off deviating by entering the race. Pick a potential candidate $h \in \mathcal{P}$ with $x_h = x_M$, and construct candidacy profile \tilde{e} such that $\tilde{e}_h = 1$ and $\tilde{e}_k = e_k$ for all $k \in \mathcal{P} \setminus \{h\}$. Letting $\alpha^n(\mathcal{C}(\tilde{e})) \equiv (\alpha_i^n, \alpha_h^n, \alpha_j^n)$ denote citizen n 's sincere vote given the set of candidates $\mathcal{C}(\tilde{e})$, we have

$$\alpha^n(\mathcal{C}(\tilde{e})) = \begin{cases} (1, 2, 3) & \text{if } x_n < \underline{x} \\ (2, 1, 3) & \text{if } x_n \in (\underline{x}, m) \\ (3, 1, 2) & \text{if } x_n \in (m, \bar{x}) \\ (3, 2, 1) & \text{if } x_n > \bar{x} \end{cases}$$

for every citizen $n \in \mathcal{N}$.¹⁸

If the election is held under the Plurality rule, candidates' vote totals are given by

$$\begin{cases} V_i(\mathcal{C}(\tilde{e}), \alpha) = F(\underline{x}) \\ V_h(\mathcal{C}(\tilde{e}), \alpha) = F(\bar{x}) - F(\underline{x}) \\ V_j(\mathcal{C}(\tilde{e}), \alpha) = 1 - F(\bar{x}). \end{cases}$$

From here it is easy to check that potential candidate h would be worse off entering the race only if one of the four conditions in (iii-Plurality rule) is satisfied.

If the election is held under the Alternative Vote rule, candidates' first-place votes are as given above under the Plurality rule. Key to observe is that $W_L = \{h\}$ if $L^1 \in \{i, j\}$. From here it is easy to check that potential candidate h would be worse off entering the race only if one of the four conditions in (iii-Alternative Vote rule) is satisfied.

¹⁸A citizen n with ideal policy $x_n \in \{\underline{x}, m, \bar{x}\}$ has two sincere votes. Those citizens are ignored here since they are of measure zero. Alternatively, one could assume that they randomize equiprobably between their two sincere votes.

Finally, it is straightforward to check that under the Plurality rule and the Alternative Vote rule, the three conditions in the statement are together sufficient for the existence of a 2-position equilibrium. ■

Proof of Lemma 4. Assume by way of contradiction that a 3-position equilibrium (e, α) exists under the Plurality rule or the Alternative Vote rule.

First, we know from Lemma 1 that there must be exactly one candidate at each of the three positions. We shall call the candidates at x_L , x_M and x_R , candidates L , M and R , respectively.

Second, we show that candidate L and/or candidate R would be better off deviating by not running. Let π_L , π_M and π_R denote the equilibrium probabilities that candidate L , candidate M and candidate R be elected, respectively. Without loss of generality suppose $\pi_L \leq \pi_R$. Observe that candidate L 's expected utility is equal to

$$\pi_L u(|x_L - x_L|) + \pi_M u(|x_L - x_M|) + \pi_R u(|x_L - x_R|) - \delta.$$

Suppose candidate L were to deviate by not running. Candidate M would then be elected outright, and candidate L 's utility would be equal to $u(|x_L - x_M|)$. Since (e, α) is an equilibrium, it must then be that

$$\pi_L u(|x_L - x_L|) + \pi_M u(|x_L - x_M|) + \pi_R u(|x_L - x_R|) - \delta \geq u(|x_L - x_M|),$$

which, together with $\delta > 0$, implies $\pi_M < 1$ and

$$\frac{\pi_L u(|x_L - x_L|) + \pi_R u(|x_L - x_R|)}{\pi_L + \pi_R} > u(|x_L - x_M|).$$

At the same time, the concavity of $u(\cdot)$ implies

$$u(|x_L - x_M|) \geq \frac{u(|x_L - x_L|) + u(|x_L - x_R|)}{2}.$$

Taken together, these two inequalities imply

$$(\pi_R - \pi_L)[u(|x_L - x_R|) - u(|x_L - x_L|)] > 0,$$

a contradiction since $\pi_R \geq \pi_L$ and $u(|x_L - x_R|) < u(|x_L - x_L|)$. ■

The next lemma establishes that under either of the two voting rules, the set of policies which adoption can be supported is an interval. It follows that the set of

polarization levels $Y^r \subseteq (0, 1/2]$ for voting rule $r \in \{PR, AVR\}$ (where PR stands for the Plurality rule and AVR for the Alternative Vote rule) is either empty or is an interval $I^r \in \{(0, y), (0, y]\}$ for some $y \in (0, 1/2]$. We use this result to prove Proposition 1.

LEMMA 5. *The set of policies which can be supported under the Plurality rule or the Alternative Vote rule is an interval centered around the median m .*

Proof of Lemma 5. It follows straightforwardly from condition (1) in Lemma 2 that for any configuration (x_L, x_M, x_R) an equilibrium (e, α) exists in which $x_M = m$ is adopted with probability one.

Consider a given configuration of positions (x_L, m, x_R) , and suppose that an equilibrium (e, α) exists in which x_L is adopted with a strictly positive probability. To prove the result, it is sufficient to show that for a given configuration (x'_L, m, x'_R) with $x'_L \in (x_L, m)$ an equilibrium (e', α') exists in which x'_L is adopted with a strictly positive probability. (A similar argument holds for x_R and x'_R .)

There are two cases to consider:

(1) $\delta > -\frac{u(|x'_L - x'_R|)}{2}$. It follows from condition (2) of Lemma 2 that a 1-position equilibrium (e', α') exists in which a candidate at x'_L runs unopposed, and x'_L is adopted with probability one.

(2) $\delta \leq -\frac{u(|x'_L - x'_R|)}{2}$. First, we observe that together $|x'_L - x'_R| < |x_L - x_R|$ and $u(\cdot)$ being a strictly decreasing function imply $-u(|x'_L - x'_R|) < -u(|x_L - x_R|)$. It follows that $\delta < -\frac{u(|x_L - x_R|)}{2}$. Given condition (2) of Lemma 2 and given Lemma 4, (e, α) must then be a 2-position equilibrium, and condition (iii) from Lemma 3 must be satisfied for configuration (x_L, m, x_R) .

Second, observe that $x'_L \in (x_L, m)$ implies $\underline{x} < \underline{x}'$ and $\bar{x} > \bar{x}'$, where

$$\begin{cases} \underline{x} \equiv \frac{x_L + m}{2} \text{ and } \underline{x}' \equiv \frac{x'_L + m}{2} \\ \bar{x} \equiv \frac{x_R + m}{2} \text{ and } \bar{x}' \equiv \frac{x'_R + m}{2}. \end{cases}$$

It follows that $F(\underline{x}) < F(\underline{x}')$, $1 - F(\bar{x}) < 1 - F(\bar{x}')$ and $F(\bar{x}) - F(\underline{x}) > F(\bar{x}') - F(\underline{x}')$. These three inequalities, together with $-u(|x_L - x_R|) > -u(|x'_L - x'_R|)$, imply that condition (iii) from Lemma 3 being satisfied for configuration (x_L, m, x_R) is satisfied as well for configuration (x'_L, m, x'_R) .

Hence, an equilibrium (e', α') exists for configuration (x'_L, m, x'_R) , in which one candidate at x'_L and one candidate at x'_R stand for election and each is elected with probability 1/2.

This completes the proof since these two cases exhaust all possibilities. ■

Proof of Proposition 1. We already know from Lemma 5 that the set of polarization levels under voting rule r , Y^r , is empty or is an interval $(0, y]$ or $(0, y)$ for some $y \in (0, 1/2]$. To compare the degrees of policy polarization that the Plurality rule and the Alternative Vote rule can support, it is therefore sufficient to compare the upper-bounds of Y^{PR} and Y^{AVR} .

It is not difficult to construct examples of communities where $Y^{PR} = Y^{AVR}$ and, therefore, where the Plurality rule and the Alternative Vote rule support as much policy polarization as the other (cfr Corollary 1). So assume from now on that $Y^{PR} \neq Y^{AVR}$.

We already know from Lemma 2 that the sets of 1-position equilibria are equivalent under both rules. Moreover, together condition (2) from Lemma 2 and condition (ii) from Lemma 3 imply that any 2-position equilibrium is more polarized than any 1-position equilibrium. Finally, we know from Lemma 4 that there are no 3-position equilibria. It follows that the upper-bound of Y^r , for $r \in \{PR, AVR\}$, is determined by condition (iii) from Lemma 3.

For a given configuration of positions (x_L, x_M, x_R) , if condition (iii-Alternative Vote rule) is satisfied, then condition (iii-Plurality rule) is satisfied as well. The converse is not necessarily true. Hence, for a given configuration (x_L, x_M, x_R) the existence of a 2-position equilibrium under the Alternative Vote rule implies the existence of a 2-position equilibrium under the Plurality rule, but the converse is not true. It follows that $Y^{AVR} \subseteq Y^{PR}$. ■

Proof of Corollary 1. We already know from Lemma 2 that the set of 1-position equilibria is equivalent whether the election is held under the Plurality rule or under the Alternative Vote rule. Also, we know from Lemma 4 that no 3-position equilibrium exists under either rule. Finally, given the assumption on $F(\cdot)$, we have that $F(\underline{x}) = 1 - F(\bar{x})$ and, therefore, that

$$\min \{F(\underline{x}), 1 - F(\bar{x})\} = \max \{F(\underline{x}), 1 - F(\bar{x})\}.$$

It follows that, except for the third part of condition (iii) (which applies only to the specific configuration of positions where $F(\underline{x}) = \frac{F(\bar{x})}{2}$), all the conditions from

Lemma 3 are equivalent under the Plurality rule and the Alternative Vote rule. Since for a given $F(\cdot)$, the third part of condition (iii) is non-generic on X , we have generically that $Y^{AVR} = Y^{PR}$. ■