Existence of Equilibrium in Incomplete Markets with Non-Ordered Preferences

Erkan Yalcin and Duygu Yengin
Existence of Equilibrium in Incomplete Markets with Non-Ordered Preferences

Erkan Yalcin†‡ Duygu Yengin§

October 5, 2010

Abstract

In this paper we extend the results of recent studies on the existence of equilibrium in finite dimensional asset markets for both bounded and unbounded economies. We do not assume that the individual’s preferences are complete or transitive. Our existence theorems for asset markets allow for short selling. We shall also show that the equilibrium achieves a constrained core within the same framework.

Keywords. Incomplete Preferences, Intransitive Preferences, Incomplete Markets, General Equilibrium, Constrained Core, Convex Analysis

1 Introduction

In this paper, we extend the study of the existence and optimality of competitive equilibria to a wider class of economies: the class of economies, suggested by Arrow (1951), in which the asset market is possibly incomplete. Recent studies on this topic take the extension of general equilibrium theory as its starting point, which is due to Arrow & Debreu (1954). The first equilibrium existence result when consumption sets are unbounded below was proven by Hart (1974) under the assumption that consumers’ utility functions were Von Neumann-Morgenstern and that their directions of improvement were positively semi-independent. Later, Werner (1987) gave an existence result under the assumption that there exists at least one price for which there are no-arbitrage opportunities for all consumers. Making fairly weak assumptions on preferences, Nielsen (1989) obtained a very general result under the assumption that consumers’ directions of improvement were positively semi-independent.

In general, arbitrage conditions are sufficient to guarantee existence of equilibria when short-sales are allowed in asset exchange economies. For instance, Dana et al. (1999) proved an equilibrium existence theorem, with consumption sets that are unbounded below, only assuming

*During the preparation of this paper we have been influenced by helpful comments with many persons. Among those who deserve special mention are Hans Haller, David Kelsey and Matthew Ryan.
†School of Business, Economics and Public Policy, University of New England, Armidale, NSW 2351, Australia (E-mail: eyalcin@une.edu.au).
‡School of Economics, University of Adelaide, SA 5005, Australia (E-mail: erkan.yalcin@adelaide.edu.au).
§School of Economics, University of Adelaide, SA 5005, Australia (E-mail: duygu.yengin@adelaide.edu.au).
that the individually rational utility set is compact. Recently, Le Van et al. (2001) introduced consumption externalities into a general equilibrium model, where consumption sets are closed, convex and possibly unbounded. They showed that a generalized condition of \textit{no unbounded arbitrage} is sufficient for the existence of equilibrium and necessary and sufficient for compactness of the set of individually rational allocations.

The aim of the paper is to generalize the previous literature in a number of directions. First, we shall show the existence of a competitive equilibrium in incomplete markets. It is well known that even in the simplest case of an economy with incomplete markets, where there is only one physical commodity\footnote{It is fairly standard to make such an assumption in the finance literature, see for instance Lintner (1965), Sharpe (1964), Milne (1988), Kelsey & Milne (1995), Bettzüge (1998) and Kelsey & Yalcin (2007).}, equilibrium will, in general, fail to exist. This paper provides the sufficient conditions on asset payoffs, preferences, and endowments under which the equilibrium of the underlying economy exists when there is only one physical commodity.

Second, we shall not assume that the preferences are complete or transitive. For instance, most investors in financial markets are not single investors but rather corporate bodies. Therefore, most investment decisions are collective decisions. If markets are complete, then all group members would have the same preferences over investments. If markets are incomplete, then it is not possible to evaluate market values of all feasible investment decisions from available price system. As a result, even if the competitive conditions prevail, generically, investors will not be unanimous over the choice of corporate investment plans, see for instance Duffie & Shafer (1985) and Haller (1991). Likewise, different investors will have different preferences over the corporate investment plans. In such cases, a corporate’s investment decision will be the outcome of a collective decision process. Social choice theory implies that the outcome of such collective decision processes may be incomplete or intransitive, if the processes are non-dictatorial.\footnote{Also, Knightian uncertainty (ambiguity) can give rise to incomplete preferences. Knight (1921) distinguishes between risk, where the probabilities are known and ambiguity, where probabilities are not known.}

Third, under weak conditions on the \textit{strict preference relations}, the existence result will be extended to economies in which unrestricted short selling of assets is allowed and hence the portfolio space is not necessarily bounded below, see for instance Milne (1976), Werner (1987), and Page & Wooders (1996). Thus, in our paper, existence is not standard since the asset consumption set $A^b$ is potentially unbounded. Under these generalizations, namely incomplete markets, non-ordered preferences, and unbounded asset sets, we prove the existence of competitive equilibrium.

Finally, we present the first fundamental theorem of welfare economics in such a framework. We shall prove that if the portfolio space of an asset exchange economy is finite dimensional and the aggregate endowment is strictly positive, then the allocation is in the \textit{Constrained Core} whenever the allocation is supported by the price system.

In the following section, we present the model. In Section 3, we prove the existence of a competitive equilibrium when portfolio space is bounded. Then, in Section 4, we prove the existence of equilibrium for the unbounded case. We also establish constrained Pareto optimality of the equilibrium. Finally, the concluding section discusses some of the implications of these
results and contains some remarks about extensions of the analysis.

2 The Economy

In this section, we analyze the properties of competitive equilibrium in the context of a finite asset exchange economy under uncertainty, where trade in assets is competitive. Economic activity occurs over two time periods, $t = 0, 1$. Uncertainty is described by states of the world, indexed by $s \in S = \{1, \ldots, S\}$, a finite and non-empty set, and is resolved all in the second period. There is only one physical commodity so that the first period commodity space is $\mathbb{R}$ and the second period contingent commodity space is $\mathbb{R}^S$ making the total commodity space $\mathbb{R}^{S+1}$. However, we shall consider in the sequel an exchange economy where second period actions by consumers are restricted to trades in assets that offer linear combinations of contingent commodities. Therefore, we shall treat the assets to be the objects of choice rather than examining the contingent commodities explicitly.

There are a finite number of consumers, indexed by $h \in H = \{1, \ldots, H\}$. Each consumer $h$ has a state-contingent commodity consumption set $X^h \subset \mathbb{R}^{S+1}$.

Each agent $h$ has a strict preference relation $\succ_h$ defined on $X^h$, that is irreflexive but may not be complete or transitive. For each $h \in H$ and each $x \in X^h$, let $U(\succ_h, x) = \{y \in X^h : y \succ_h x\}$ and $L(\succ_h, x) = \{y \in X^h : x \succ_h y\}$ be the strict upper contour set and strict lower contour set of $x \in X^h$ with respect to the preference relation $\succ_h$, respectively.

Consider the following assumptions on $X^h$ and preferences defined on $X^h$.

**Assumption 2.1**

(a) For each $h \in H$, $X^h$ is non-empty, closed, convex, and bounded below.

(b) Irreflexivity: For each $h \in H$ and each $x \in X^h$, $x \notin U(\succ_h, x)$.

(c) Nonsatiation: For each $h \in H$ and each $x \in X^h$, $x \in \text{cl}(U(\succ_h, x))$, where “cl” stands for the “closure”.

(d) Continuity: For each $h \in H$ and each $x \in X^h$, the sets $U(\succ_h, x)$ and $L(\succ_h, x)$ are open subsets of $X^h$.

(e) Convexity: For each $h \in H$ and each $x \in X^h$, the set $U(\succ_h, x)$ is non-empty and convex.\(^3\)

2.1 Induced Preferences

Let there be $J$ assets indexed by $j \in J = \{1, \ldots, J\}$. Consumer preferences on $X^h$ generates derived preferences over asset holdings. We shall refer to the latter as induced preferences.

Define the commodity space in the asset economy to be the space $\mathbb{R}^{J+1}$, where there are $J$ assets and the first period commodity (asset 0). One unit of the $j$–th asset pays a pattern of returns in contingent commodities, $Z_j = (Z_{ja})_{s \in S} \in \mathbb{R}^S$. For each $h \in H$, let $a^h \in \mathbb{R}^{J+1}$ be the vector of asset holdings of consumer $h$ where $a^h_j$ defines the number of the $j$th asset held by

\(^3\)We have assumed, without loss of generality, that $U(\succ_h, x)$ is convex and $x \notin U(\succ_h, x)$. Suppose not, then we can replace $\succ^h: X^h \to X^h$ by $\succ^h: X^h \to X^h$, where $\hat{U}(\succ_h, x) = \text{con}(U(\succ_h, x))$ where con indicates convex hull. The binary relation in question will still have open graph and by Assumption 2.1, $x \notin \hat{U}(\succ_h, x)$ (see Border (1984)).
consumer $h$. Let $Z = [Z_1, Z_2, \ldots, Z_J]$ be the $S \times J$ matrix representing the market set of returns.

In order to derive consumer preferences over assets, let us define a function $\Lambda : \mathbb{R}^{J+1} \rightarrow \mathbb{R}^{S+1}$ by $Z'\beta = \alpha$, where $\beta \in \mathbb{R}^{J+1}$, $\alpha \in \mathbb{R}^{S+1}$, and $Z'$ is the $(S+1) \times (J+1)$ semi-positive matrix such that

$$Z' = \begin{bmatrix} 1 & 0 \\ 0 & Z \end{bmatrix}.$$  

The function $\Lambda$ is linear and onto the range $Q$, which is a vector subspace of dimension $(J + 1)$. Define $V^h = Q \cap X^h$, where $V^h \neq \emptyset$ since $\{0\} \subset Q \cap X^h$. Define consumer $h$’s consumption set in asset economy (feasible portfolio space) $A^h$ by $\Lambda^{-1} : V^h \rightarrow \mathbb{R}^{J+1}$, that is, $A^h \equiv \Lambda^{-1} (V^h)$.

Using $\Lambda^{-1}$, induced preferences $\succsim^h$ over assets can be derived from commodity preferences $\succsim_h$. In other words, assets are desired solely for the returns they yield in the second period, therefore, preferences over assets are derived preferences. The next Lemma presents the properties of induced preferences based on Assumption 2.1.

For each $h \in H$ and each $a^h \in A^h$, let $U(\succsim^h_h, a^h) = \{ \hat{a}^h \in A^h : \hat{a}^h \succsim^h_h a^h \}$ and $L(\succsim^h_h, a^h) = \{ \hat{a}^h \in A^h : a^h \succsim^h_h \hat{a}^h \}$. 

**Lemma 2.2** Let Assumption 2.1 be satisfied and the following condition hold: for each $x \in V^h$, $\exists y \in V^h$ such that $y \succsim_h x$.

Then, for each $h \in H$, 

(a) $A^h$ is non-empty, closed, convex, and bounded below$^4$; 
(b) Irreflexivity: for each $a^h \in A^h$, $a^h \notin U(\succsim_h, a^h)$; 
(c) Nonsatiation: for each $a^h \in A^h$, $a^h \in cl(U(\succsim_h, a^h))$, where “cl” stands for the “closure”; 
(d) Continuity: for each $a^h \in A^h$, the sets $U(\succsim^h_h, a^h)$ and $L(\succsim^h_h, a^h)$ are open subsets of $A^h$; 
(e) Convexity: for each $a^h \in A^h$, the set $U(\succsim^h_h, a^h)$ is non-empty and convex.

**Proof.** The proof of Lemma 2.2 can be found in Milne (1976) (Lemma 1) which was given for the weak preference relation. In our case, this involves a trivial modification for the strict preference relation. Therefore, we will omit the proof. $\blacksquare$

### 3 Equilibria in Bounded Economies

In this section, we wish to prove the existence of a competitive equilibrium in an economy where arbitrary bounds are imposed on trades. Then, in Section 4, we prove existence in general by letting these bounds tend to infinity.

Let $E = (A^h, \bar{\pi}^h, \succsim^h_h)_{h \in H}$ be an asset exchange economy in which each consumer $h$ has an asset consumption set $A^h$, an initial endowment of assets $\bar{\pi}^h \in \mathbb{R}^{J+1}$, and a preference relation $\succsim^h_h$ on $A^h$. Consumer $h$ can trade $\bar{\pi}^h$ to obtain a new portfolio of assets. Let $q \in \mathbb{R}^{J+1} \setminus \{0\}$ be

$^4$The assumption that each $A^h$ is closed and bounded below can be replaced by the assumption that each $A^h$ is compact, which is standard, see Debreu (1959).
the price vector for trades in asset exchange economy. Let consumer $h$’s budget set for a given price system $q \in \mathbb{R}^{J+1} \setminus \{0\}$ be

$$B^h(q) = \left\{ a^h \in A^h : q \cdot a^h \leq q \cdot \bar{a}^h \right\}. $$

Let

$$\Omega = \left\{ (a^h)_{h \in \mathcal{H}} \in \prod_{h \in \mathcal{H}} A^h : \sum_{h \in \mathcal{H}} a^h = \sum_{h \in \mathcal{H}} \bar{a}^h \right\}$$

be the attainable state of the economy.

**Definition 3.1** An equilibrium for an asset economy $E = (A^h, a^h, \succ_h)_{h \in \mathcal{H}}$ is a collection $(a^*, q^*)$ of asset holdings $a^* = (a^h)_{h \in \mathcal{H}} \in \Omega$ and prices $q^* \in \mathbb{R}^{J+1} \setminus \{0\}$ such that for each $h \in \mathcal{H},$

a) $a^h \in B^h(q^*);$

b) $U(\succ_h, a^h) \cap B^h(q^*) = \emptyset.$

To prove the existence of a competitive equilibrium for any given bounded asset exchange economy $E = (A^h, a^h, \succ_h)_{h \in \mathcal{H}},$ first, we will construct an associated economy $\hat{E} = (\hat{A}^h, \hat{a}^h, \hat{\succ}_h)_{h \in \mathcal{H}}$ as described below. Then, we will show that by Shafer (1976), a competitive equilibrium $(\hat{a}^*, \hat{q}^*)$ exists for economy $\hat{E}.$ Finally, we will prove that this equilibrium is also an equilibrium for $E.$

Since, for each $h \in \mathcal{H}, A^h$ is bounded below, let $b \in \mathbb{R}^{J+1}$ be such that for each $h \in \mathcal{H}$ and each $a^h \in A^h,$

$$b < \min_{\mathcal{H}' \subseteq \mathcal{H}} \{ \sum_{h \in \mathcal{H}'} a^h \}. \quad (3.1)$$

Let $\bar{b} \in \mathbb{R}^{J+1}$ be such that

$$\sum_{h \in \mathcal{H}} a^h < \bar{b}. \quad (3.2)$$

For each $h \in \mathcal{H},$ let

$$\hat{A}^h = \left\{ a^h \in A^h : a^h \leq \bar{b} - b \right\}. $$

Let $\hat{A} = \prod_{h \in \mathcal{H}} \hat{A}^h.$

**Lemma 3.2** If $a = (a^h)_{h \in \mathcal{H}} \in \hat{A} \cap \Omega,$ then for each $h \in \mathcal{H},$ $a^h \ll \bar{b} - b.$

**Proof.** Assume, by contradiction, that there is $\hat{h} \in \mathcal{H}$ and $j \in \mathcal{J}$ such that $a^j_{\hat{h}} = \bar{b}_j - b_j.$ Then,

$$\sum_{h \in \mathcal{H}} \bar{a}^j = a^j_{\hat{h}} + \sum_{h \in \mathcal{H} \setminus \{\hat{h}\}} a^j_h = \bar{b}_j - b_j + \sum_{h \in \mathcal{H} \setminus \{\hat{h}\}} a^j_h > \bar{b}_j \quad (3.3)$$

where the first equality follows from the fact that $a \in \Omega$ and the last inequality follows from (3.1). By (3.3), $\sum_{h \in \mathcal{H}} a^j_h > \bar{b}_j,$ which contradicts (3.2). $lacksquare$

By Lemma 3.2, we have the following result:

**Corollary 3.3** If $a = (a^h)_{h \in \mathcal{H}} \in \hat{A} \cap \Omega,$ then $a \in \text{int}\hat{A}$ and for each $h \in \mathcal{H}, a^h \in \text{int}\hat{A}^h.$
We will only sketch the proof of the existence of an equilibrium for \( \hat{\mathcal{E}} \) since it is standard. For each \( h \in \mathcal{H} \) and each \( a^h \in \hat{A}^h \), let
\[
P_h \left( >_h, a^h \right) = \left\{ \tilde{a}^h \in A^h : \tilde{a}^h = \lambda \tilde{a}^h + (1 - \lambda) a^h \text{ for } 0 < \lambda \leq 1 \text{ and } \tilde{a}^h \in U(>_h, a^h) \right\}.
\]
For each \( h \in \mathcal{H} \), let \( \widehat{\mathcal{A}}^h \) be the asset preference of agent \( h \) associated with the upper contour sets where for each \( a^h \in \hat{A}^h \), \( U(\widehat{\mathcal{A}}^h, a^h) = P_h \left( >_h, a^h \right) \). The economy \( \hat{\mathcal{E}} = \left( \hat{A}^h, \pi^h, \widehat{\mathcal{A}}^h \right)_{h \in \mathcal{H}} \) so constructed satisfies the following conditions: For each \( h \in \mathcal{H} \),

i. \( \hat{A}^h \) is non-empty, convex, and compact;

ii. \( \pi^h \in \text{int} \hat{A}^h ; \)

iii. For each \( a^h \in \hat{A}^h \), the sets \( U(\widehat{\mathcal{A}}^h, a^h) \) and \( L(\widehat{\mathcal{A}}^h, a^h) \) are open subsets of \( \hat{A}^h \);

iv. For each \( a^h \in \hat{A}^h \), the set \( U(\widehat{\mathcal{A}}^h, a^h) \) is non-empty and convex;

v. For each \( a = (a^h)_{h \in \mathcal{H}} \in \hat{\mathcal{A}} \cap \Omega \) and each \( h \in \mathcal{H} \), \( a^h \in \partial (U(\widehat{\mathcal{A}}^h, a^h)) \), where “\( \partial \)” stands for the “boundary”.

Therefore, by Shafer (1976), \( \hat{\mathcal{E}} \) has an equilibrium \( (\hat{a}^*, \hat{q}^*) \), that is, \( \hat{a}^* \in \Omega \), \( \hat{q}^* \in \mathbb{R}^{t+1} \setminus \{0\} \), and for each \( h \in \mathcal{H} \),

a) \( \hat{a}^{*h} \in \hat{B}^h (q^*) = \left\{ a^h \in \hat{A}^h : q^* \cdot a^h \leq q^* \cdot \pi^h \right\} ; \)

b) \( U(\hat{\mathcal{A}}^h, \hat{a}^{*h}) \cap \hat{B}^h (q^*) = \emptyset . \)

Next, we show that if for each \( h \in \mathcal{H} \), \( A^h \) and \( >_h \) satisfy the conditions listed in Lemma 2.2, then a bounded asset exchange economy \( \mathcal{E} = (A^h, \pi^h, >_h)_{h \in \mathcal{H}} \) has a competitive equilibrium that coincides with the equilibrium of economy \( \hat{\mathcal{E}} = \left( \hat{A}^h, \pi^h, \hat{a}^h \right)_{h \in \mathcal{H}} \).

**Theorem 3.4** Equilibrium \( (\hat{a}^*, \hat{q}^*) \) is also a competitive equilibrium for \( \mathcal{E} = (A^h, \pi^h, >_h)_{h \in \mathcal{H}} \).

**Proof.** Clearly, for each \( h \in \mathcal{H} \), \( \hat{a}^* \in \hat{B}^h (q^*) \subseteq B^h (q^*) \). Therefore, it is sufficient to show that for each \( h \in \mathcal{H} \), \( U(\hat{a}^{*h}) \cap B^h (q^*) = \emptyset . \)

Assume, by contradiction, that there is \( h \in \mathcal{H} \) such that \( U(\hat{a}^{*h}) \cap B^h (q^*) \neq \emptyset . \)

Let \( \tilde{a}^h \in U(\hat{a}^{*h}) \cap B^h (q^*) \). Then, for each \( 0 < \lambda < 1 \), define \( a^h = 1 - \lambda \tilde{a}^h + (1 - \lambda) \hat{a}^{*h} \). By definition, \( a^h \in P_h \left( \hat{a}^{*h} \right) \).

Since \( \{\tilde{a}^{*h}, \tilde{a}^h\} \subseteq B^h (q^*) \), we have \( q^* \cdot \tilde{a}^{*h} \leq q^* \cdot \pi^h \) and \( q^* \cdot \tilde{a}^h \leq q^* \cdot \pi^h \). Then, for each \( 0 < \lambda < 1 \),
\[
q^* \cdot a^h \leq q^* \cdot \pi^h . \quad (3.4)
\]

Since \( \hat{a}^{*h} \in \hat{A} \cap \Omega \), by Corollary 3.3, \( \hat{a}^{*h} \in \text{int} \hat{A}^h \). Therefore, there exists sufficiently small \( \lambda \) such that \( a^h \in \hat{A}^h \), which by (3.4), implies that \( \hat{a}^h \in B^h (q^*) \). Then, \( a^h \in U(\hat{a}^{*h}) \cap B^h (q^*) \), which contradicts condition (b) for \( (\hat{a}^*, \hat{q}^*) \) to be an equilibrium for \( \hat{\mathcal{E}} \). This completes the proof. 

\( \blacksquare \)
4 Equilibria in Unbounded Economies

The fact that we treat assets as claims to contingent consumption in the second period has an important effect on the problem of the existence of competitive equilibria. In this section, we will allow for the possibility that consumers can go arbitrarily short in asset trading. Since consumers are allowed to sell assets short, we will work with portfolio spaces without a prior lower bound. Thus, we shall provide a basic result that shows the existence of a competitive equilibrium in an economy with unbounded asset trade sets.

For each \( h \in H \), let \( X^h = R^{S+1} \) be the consumption possibility set of consumer \( h \). As before, a portfolio of assets is defined as a vector \( a^h \in R^{J+1} \). We shall assume that \( a^h \) may be positive or negative. Define \( sp(Z') \) to be the span of \( (Z') \). In the presence of asset markets with an incomplete structure, the consumption set of each consumer \( h \) can be specified as follows:

\[
X^h = X^h \cap \left\{ x^h \in R^{S+1} : x^h \in sp(Z') \right\} \subset R^{S+1}
\]

that is, \( X^h \) is the set of contingent-commodity bundles attainable by way of the exchange of assets. Asset markets so constructed may be incomplete in the sense that the available assets do not span \( X^h \). For each consumer \( h \in H \), let the consumption set in the unbounded asset economy be as follows:

\[
A^h = \left\{ a^h \in R^{J+1} : Z'a^h \in X^h \right\},
\]

where \( Z' \) is a semi-positive matrix defined in (2.1). Let \( A = \prod_{h \in H} A^h. \)

Notice that \( A^h \) is assumed to have no lower bound. Also, note that since \( X^h = R^{S+1} \) for each \( h \in H \) and \( sp(Z') \) is same for all agents, for each pair \( \{h, h'\} \subseteq H \), \( X^h = X^{h'} \) and hence, \( A^h = A^{h'}. \)

Assumption 4.1 For each \( h \in H \), the initial asset endowment of \( h, \bar{a}^h \in R^{J+1} \) is in the interior of \( A^h \), that is, \( \bar{a}^h \in int A^h. \)

We need the following definition:

Definition 4.2 Given a set \( X \), we say that \( d \in X \) is a direction of recession for \( X \) if for any \( x \in X \), the ray \( \{ x + \lambda d : \lambda \geq 0 \} \) is also in \( X \).

Let \( O^+X = \{ d : x + \lambda d \in X, x \in X, \lambda \geq 0 \} \) be the set of all recession directions of \( X \). \(^5\)

Alternatively, \( O^+X = \{ y : X + y \subset X \} \).

If \( X \) is a closed convex set, then \( O^+X \) is a closed convex cone containing the origin. Note that \( X \) is closed and bounded if and only if \( O^+X = \{0\} \).

Therefore, for each \( h \in H \) and each \( A^h \), the recession cone \( O^+A^h \) corresponding to \( A^h \) is a closed convex cone containing the origin.

Since each unit of asset \( j \in J \) is a contract that promises to pay a fixed non-negative vector \( Z_j \in R^S \), and assuming that consumer \( h \) has no other source of wealth in the second period, one can obtain the following result.

\(^5\)See Section 8 in Rockafellar (1970).
Lemma 4.3 Assume that rank \((Z') = J + 1\) and for each \(h \in \mathcal{H}\), \(X^h = \mathbb{R}^{S+1}_+\). Then, the derived asset set satisfies the following condition: for each \(\{h, h'\} \subseteq \mathcal{H}\),

\[
O^+ A^h \cap O^+ (\neg A^{h'}) = \{0\}.
\]

Proof. By definition, for each \(h \in \mathcal{H}\) and each \(a^h \in A^h\), \(Z'a^h \geq 0\). Hence,

\[
-A^h = \left\{a^h \in \mathbb{R}^{J+1} : Z'a^h \leq 0, \ Z' \text{ semi-positive, } -a^h \in A^h\right\}.
\]

Hence, if \(a^h \in A^h \cap (\neg A^h)\), then \(Z'a^h = 0\). This equality and the fact that rank \((Z') = J + 1\) together imply \(a^h = 0\). Hence, (i) \(A^h \cap (\neg A^h) = \{0\}\). Note that (ii) \(0 \in [O^+ A^h \cap O^+ (\neg A^h)]\).

Since \(O^+ A^h \subset A^h\) and \(O^+ (\neg A^h) \subset A^h\), then (iii) \(O^+ A^h \cap O^+ (\neg A^h) \subset A^h \cap (\neg A^h)\).

Hence, (i), (ii), and (iii) together imply that for each \(h \in \mathcal{H}\), \(O^+ A^h \cap O^+ (\neg A^h) = \{0\}\). Since for each pair \(\{h, h'\} \subseteq \mathcal{H}\), \(A^h = A^{h'}\), we have the desired result. \(\blacksquare\)

Let \(\mathcal{E}^u = (A^h, >_h, \pi^h)_{h \in \mathcal{H}}\) denote the unbounded asset exchange economy where each consumer \(h \in \mathcal{H}\) has an asset consumption set \(A^h \subset \mathbb{R}^{J+1}\), preferences \(>_h\) over \(A^h\), and an endowment of assets \(\pi^h \in A^h\). Consumer \(h\)'s preferences over \(A^h\) are specified by a strict preference relation \(>_h\) associated with upper and lower contour sets where for each \(a^h \in A^h\), \(U(>_h, a^h) = \{\hat{a}^h \in A^h : \hat{a}^h \succ_h a^h\}\) and \(L(>_h, a^h) = \{\hat{a}^h \in A^h : a^h \succ_h \hat{a}^h\}\). Throughout, we shall assume that for each \(a^h \in A^h\), \(U(>_h, a^h)\) exhibits the following properties:

**Assumption 4.4** For each \(h \in \mathcal{H}\) and each \(a^h \in A^h\),

(a) \(a^h \notin U(>_h, a^h)\);

(b) \(a^h \in cl(U(>_h, a^h))\), where “cl” stands for the “closure”\(^6\);

(c) The sets \(U(>_h, a^h)\) and \(L(>_h, a^h)\) are open subsets of \(A^h\);

(d) The set \(U(>_h, a^h)\) is non-empty and convex.

Let \(Q^u = \{(a^h)_{h \in \mathcal{H}} \in A : \sum_{h \in \mathcal{H}} a^h = \sum_{h \in \mathcal{H}} \pi^h\}\) be the set of attainable economies when asset sets are unbounded. Let \(Q = \{q \in \mathbb{R}^{J+1} : \|q\| \leq 1\}\) be the set of relative prices. Note that \(Q\) is compact. Let \(B^h(q) = \{a^h \in A^h : q \cdot a^h \leq q \cdot \pi^h\}\).

**Definition 4.5** An equilibrium for an unbounded economy \(\mathcal{E}^u = (A^h, \pi^h, >_h)_{h \in \mathcal{H}}\) is a collection \((a^*, q^*)\) of asset holdings \(a^* = (a^h)_{h \in \mathcal{H}}\) and prices \(q^* \in Q\) such that for each \(h \in \mathcal{H}\),

(a) \(a^h \in B^h(q^*)\);

(b) \(U(>_h, a^h) \cap B^h(q^*) = \emptyset\).

We need the following definition which follows from Debreu (59).

**Definition 4.6** Cones \(A^1, ..., A^H\) (with vertex 0) are said to be positively semi independent if \(\sum_{h=1}^H a^h = 0\) and for each \(h \in \mathcal{H}\), \(a^h \in A^h\) together imply that for each \(h \in \mathcal{H}\), \(a^h = 0\). Obviously, two cones \(A^h\) and \(A^{h'}\) with vertex 0 are positively semi independent if and only if \(A^h \cap A^{h'} = \{0\}\).

\(^6\)Assumption 4.4(b) says that \(a^{h'} \succ_h a^h\) implies \(\lambda a^{h'} + (1 - \lambda) a^h \succ_h a^h\) for all \(\lambda \in (0, 1]\).
Clearly, the set of attainable states \( \Omega^u \) of the asset exchange economy is closed and convex. Since for each \( h \in \mathcal{H} \), \( \mathcal{A}^h \) may be unbounded, \( \mathcal{A} \) may be unbounded. To show that \( \Omega^u \) is bounded, by Definition 4.2, it is sufficient to prove that \( O^+\Omega^u = \{0\} \).

**Proposition 4.7** Given an unbounded economy \( \mathcal{E}^u \), the set \( \Omega^u \) of attainable states is bounded.

**Proof.** Define \( O^+\Omega^u = \{ a \in \mathbb{R}^{H(J+1)} : \Omega^u + a \subset \Omega^u \} \). Let \( a = (a^h)_{h \in \mathcal{H}} \in O^+\Omega^u \) and \( \hat{a} = (\hat{a}^h)_{h \in \mathcal{H}} \in \Omega^u \). Since \( O^+\Omega^u \subset \Omega^u \), then \( a + \hat{a} \in \Omega^u \). Hence, summing over \( h \), one has \( \sum_{h \in \mathcal{H}} (a^h + \hat{a}^h) = \sum_{h \in \mathcal{H}} \bar{a}^h \). Since \( \sum_{h \in \mathcal{H}} \bar{a}^h = \sum_{h \in \mathcal{H}} (a^h + \bar{a}^h) = \sum_{h \in \mathcal{H}} \bar{a}^h \). Hence, \( \sum_{h \in \mathcal{H}} a^h = 0 \). This implies that

\[
O^+\Omega^u = \left\{ (a^h)_{h \in \mathcal{H}} \in \mathbb{R}^{H(J+1)} : a^h \in \mathbb{R}^{J+1} \quad \forall h, \quad \sum_{h \in \mathcal{H}} a^h = 0 \right\}.
\]

Next, we show that \( O^+\mathcal{A}^1, ..., O^+\mathcal{A}^H \) are positively semi-independent. Note that for each \( h \in \mathcal{H} \), \( O^+\mathcal{A}^h \) is a cone containing origin. Also, by Lemma 4.3, for each pair \( \{h, h'\} \subseteq \mathcal{H} \), \( O^+\mathcal{A}^h \cap (-O^+\mathcal{A}^{h'}) = \{0\} \). Hence, by Definition 4.6, for each pair \( \{h, h'\} \subseteq \mathcal{H} \), cones \( O^+\mathcal{A}^h \) and \(-O^+\mathcal{A}^{h'}\) are positively semi-independent. In other words, (i) \( O^+\mathcal{A}^1, ..., O^+\mathcal{A}^H \) are positively semi-independent.

Finally, let \( a = (a^h)_{h \in \mathcal{H}} \in O^+\Omega^u \) which implies (ii) \( \sum_{h \in \mathcal{H}} a^h = 0 \). Since \( \Omega^u \subset \mathcal{A} \), then \( a \in O^+\Omega^u \subset O^+\mathcal{A} \subset \prod_{h \in \mathcal{H}} (O^+\mathcal{A}^h) \). This implies that (iii) for each \( h \in \mathcal{H} \), \( a^h \in O^+\mathcal{A}^h \). By Definition 4.6, (i), (ii), and (iii) together imply that for each \( h \in \mathcal{H} \), \( a^h = 0 \). Therefore \( O^+\Omega^u = \{0\} \). By Definition 4.2, \( \Omega^u \) is bounded. This completes the proof. \( \blacksquare \)

Consider a compact economy \( \hat{\mathcal{E}}_n = (\hat{\mathcal{A}}^h_n, \hat{\pi}^h_n, \succsim^a_n)_{h \in \mathcal{H}} \) with \( n \geq 1 \), such that for each \( h \in \mathcal{H} \),

1. and each \( n \geq 1 \), \( \hat{\mathcal{A}}^h_n \subset \hat{\mathcal{A}}^h_{n+1} ; 
2. \lim_{n \to \infty} \hat{\mathcal{A}}^h_n = \mathcal{A}^h ; 
3. \pi^h \in int\hat{\mathcal{A}}_n^h ; 
4. \Omega^u \subset \prod_{h \in \mathcal{H}} \hat{\mathcal{A}}^h_n.

By Shafer & Sonnenschein (1975), for each \( n \geq 1 \), there exists \( (a^*_n, q^*_n) \in \Omega^u \times \mathcal{Q} \) which is an equilibrium of \( \hat{\mathcal{E}}_n \). Hence, we have the equilibrium sequence \( \{a^*_n, q^*_n\}_{n=1}^\infty \subset \Omega^u \times \mathcal{Q} \). Note that \( \Omega^u \) is closed and by Proposition 4.7 it is bounded. Since \( \Omega^u \times \mathcal{Q} \) is compact, \( \{a^*_n, q^*_n\}_{n=1}^\infty \) has a converging subsequence. Let \( (a^*, q^*) \) be the limit of this subsequence, that is, \( \lim_{n \to \infty} (a^*_n, q^*_n) = (a^*, q^*) \).

**Theorem 4.8** The pair \( (a^*, q^*) \) is a competitive equilibrium for the unbounded economy \( \mathcal{E}^u = (\mathcal{A}^h, \pi^h, \succsim^a)_{h \in \mathcal{H}} \).

**Proof.** Note that since \( \Omega^u \times \mathcal{Q} \) is compact, \( (a^*, q^*) = \lim_{n \to \infty} (a^*_n, q^*_n) \), \( \{a^*_n\}_{n=1}^\infty \subset \Omega^u \), and \( \{q^*_n\}_{n=1}^\infty \subset \mathcal{Q} \), then \( a^* \in \Omega^u \) and \( q^* \in \mathcal{Q} \).
First, we show that, for each \( h \in \mathcal{H} \), \( a^h \in B^h (q^*) \). Suppose, by contradiction, that there is \( h \in \mathcal{H} \) such that \( a^h \notin B^h (q^*) \), that is, \( q^* \cdot a^h > q^* \cdot \overline{p}^h \). Since \( (a^*, q^*) = \lim_{n \to \infty} (a^*_n, q^*_n) \), for \( n \) sufficiently large, one must have \( q^*_n \cdot a^*_n > q^*_n \cdot \overline{p}^h \), a contradiction to the fact that \( (a^*_n, q^*_n) \) is an equilibrium for \( \hat{E}_n \).

Next, we show that for each \( h \in \mathcal{H} \), \( U(\succ_h^a, a^h) \cap B^h (q^*) = \emptyset \). Let \( h \in \mathcal{H} \).

a) Let \( \hat{a}^h \in A^h \) be such that \( q^* \cdot \hat{a}^h < q^* \cdot \overline{p}^h \). This implies, for sufficiently large \( n \), that \( \hat{a}^h \in A^h_n \) and \( q^*_n \cdot \hat{a}^h \leq q^*_n \cdot \overline{p}^h \). Hence, \( \hat{a}^h \in B^h (q^*_n) \). Since \( (a^*_n, q^*_n) \) is an equilibrium for \( \hat{E}_n \), then \( \hat{a}^h \notin U(\succ_h^a, a^h) \). Since for each \( n \geq 1 \), \( A^h_n \subset A^h_{n+1} \) and \( \lim_{n \to \infty} A^h_n = A^h \), then \( \hat{a}^h \notin U(\succ_h^a, a^h) \).

b) Now take any point \( a^h \in B^h (q^*) \). Note that \( a^h \) can be approximated by a sequence \( \{ \hat{a}^h_n \}_{n=1}^{\infty} \subset A^h \) such that \( q^* \cdot \hat{a}^h_n < q^* \cdot \overline{p}^h \) for all \( n \geq 1 \) and \( \lim_{n \to \infty} \hat{a}^h_n = a^h \).

For sufficiently large \( n \), \( \hat{a}^h_n \in A^h \). Hence, by part (a), \( q^* \cdot \hat{a}^h_n < q^* \cdot \overline{p}^h \) which implies \( \hat{a}^h \notin U(\succ_h^a, a^h) \). Since \( \lim_{n \to \infty} \hat{a}^h_n = a^h \), then \( a^h \notin U(\succ_h^a, a^h) \). Therefore, \( U(\succ_h^a, a^h) \cap B^h (q^*) = \emptyset \). This completes the proof. \( \blacksquare \)

4.1 Optimality of Competitive Allocations

There is no reason to expect an equilibrium allocation to be Pareto optimal. In fact, it is Pareto optimal, in general, only if the market structure is essentially complete. Next, we shall give a definition of Pareto optimality which is a special case of what is referred to as Constrained Core, since it only excludes Pareto improvement brought by exchanging the existing assets. Note that an element of the constrained core is constrained Pareto optimal.

**Definition 4.9** A Constrained Core with respect to the preference profile \( (\succ_h^a)_{h \in \mathcal{H}} \) is an asset portfolio allocation \( (a^h)_{h \in \mathcal{H}} \) such that there does not exist an allocation \( (\hat{a}^h)_{h \in \mathcal{H}} \) and a non-empty subset \( \mathcal{I} \subset \mathcal{H} \) such that for each \( h \in \mathcal{I} \), \( a^h \in U(\succ_h^a, a^h) \) and \( \sum_{h \in \mathcal{I}} a^h = \sum_{h \in \mathcal{I}} \overline{p}^h \).

Next result adapts the first fundamental theorem of welfare to the asset exchange economy setting we study.

**Theorem 4.10** In an asset exchange economy \( (A^h, \succ_h^a, \overline{p}^h)_{h \in \mathcal{H}} \), every competitive equilibrium \( (a^*, q^*) \) is in the Constrained Core.

**Proof.** Let \( (a^*, q^*) \) be an equilibrium allocation and price system for \( (A^h, \succ_h^a, \overline{p}^h)_{h \in \mathcal{H}} \). Suppose that there exists a non-empty subset \( \mathcal{I} \subset \mathcal{H} \) and an allocation \( (\hat{a}^h)_{h \in \mathcal{H}} \) such that \( \sum_{h \in \mathcal{I}} a^h = \sum_{h \in \mathcal{I}} \overline{p}^h \) and for each \( h \in \mathcal{I} \), \( a^h \in U(\succ_h^a, a^h) \). For each \( h \in \mathcal{I} \), since \( U(\succ_h^a, a^h) \cap B^h (q^*) = \emptyset \), then \( a^h \notin B^h (q^*) \), that is, \( q^* \cdot a^h > q^* \cdot \overline{p}^h \). Summing over \( h \in \mathcal{I} \), we get \( q^* \sum_{h \in \mathcal{I}} a^h > q^* \sum_{h \in \mathcal{I}} \overline{p}^h \) which contradicts \( \sum_{h \in \mathcal{I}} a^h = \sum_{h \in \mathcal{I}} \overline{p}^h \). \( \blacksquare \)

5 Conclusion

In this paper, we have given simple and direct equilibrium existence results for an asset exchange economy when unlimited short selling was allowed. We assumed that consumer preferences were
given by an irreflexive binary relation with open graph, that preferences were possibly incomplete or intransitive, and that the portfolio space was non-compact and finite dimensional. Our study therefore generalizes various results in the existing literature of economic theory.

Some comments are in order. First of all, in the proof of existence for an unbounded economy, it was assumed that there is an independent set of asset returns. This assumption ensures the result of Lemma 4.3, and rules out the possibility of a consumer taking an unbounded position in dependent assets. With dependent assets, it is reasonable for the consumer to issue a set of dependent assets that give the same returns as another asset held long, without violating contractual feasibility, see Milne (1976). In general, a dependent asset equilibrium can easily be derived from an independent asset equilibrium by taking appropriate linear combination of quantities and prices of independent assets, see Milne (1988) and Kelsey & Yalcin (2007).

Second, since the asset market is possibly incomplete and has a competitive equilibrium, it follows that the asset economy achieves a Pareto Optimal allocation of resources which coincides with the notion of a Constrained Optimum due to Diamond (1967).

Finally, the obvious limitation of the model is that the analysis has been restricted to a one-physical commodity case. Inclusion of many commodities would introduce the possibility of commodity price uncertainty in the second period. Despite this restriction, we believe that the model provides some useful implications for the pure theory of financial markets before more complicated assumptions are introduced (see, for instance, Bettzüge (1998)).

References


