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No Envy and Egalitarian-Equivalence under Multi-Object-Demand for Heterogeneous Objects

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No-Envy and Egalitarian-Equivalence under Multi-Object-Demand for Heterogeneous Objects

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Abstract

We study the problem of allocating heterogeneous indivisible tasks in a multi-object-demand model (i.e., each agent can be assigned multiple objects) where monetary transfers are allowed. Agents' costs for performing tasks are their private information and depend on what other tasks they are obtained with. First, we show that when costs are unrestricted or superadditive, then there is no *envy-free* and *egalitarian-equivalent* mechanism that assigns the tasks efficiently. Then, we characterize the class of *envy-free* and *egalitarian-equivalent* Groves mechanisms when costs are subadditive. Finally, within this class, we identify the Pareto-dominant subclass under a bounded-deficit condition. We show that the mechanisms in this subclass are Pareto-undominated by any other Groves mechanism or neutral Weighted-Groves mechanism satisfying the same bounded-deficit condition.
JEL Classifications: C79, D61, D63.
Key words: *egalitarianism, egalitarian-equivalence, no-envy, strategy-proofness, population monotonicity, fair division, the Groves mechanisms, the Weighted-Groves mechanisms, allocation of indivisible goods and money, multi-object-demand, discrete public goods.*

1 Introduction

A finite number of heterogeneous tasks (indivisible bads) are to be allocated among a finite set of agents who are collectively responsible for the completion of those tasks and have possibly different talents/costs to accomplish the tasks. Each agent has a cost function that specifies her cost for performing each set of tasks and these cost functions are private information of the agents. A "center" (government, jurisdictional authority, etc.) needs to allocate the tasks based on the reported costs of the agents. Monetary transfers are carried out to ensure fairness and incentive compatibility. Agents have quasilinear preferences over sets of tasks and money. One can interpret cost function of an agent as an indicator of her skill: the lower the costs a person generates to perform tasks, the more skilled she is in performing those tasks. A more skilled agent would enjoy a higher welfare from the same bundle of tasks and monetary transfer than a less skilled agent. Examples to the problem we study include the imposition of tasks as in government requisitions and eminent domain proceedings[1] and allocation of discrete public bads to localities[2] such as

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[1] Government requisition is government's demand to use goods and services of the civilians usually in times of national emergency such as natural disasters and wars. Eminent domain is government's right to seize private property, without the owner's consent, for public use such as to build a road or a public utility. In both cases, owners are legally guaranteed to receive fair monetary compensation.

[2] Assume that there is no question of whether or how much of the public good/bad is to be provided (e.g., building a waste disposal site, siting state capitals). The only question is which locality will provide what public good and what the compensations are.

the siting problem of noxious facilities (prisons, chemical process facilities, waste disposal sites, nuclear facilities etc.), also called as the "not-in-my-backyard (NIMBY)" problem. Our model and results can also be directly adapted to problems where heterogeneous indivisible goods are allocated among agents based on their reported preferences/benefits.[3] We will refer to indivisible goods and indivisible bads as objects.

The primary literature to which our paper contributes is the problem of allocating objects when monetary transfers are possible. In general, except for few papers (Beviá, 1998; and Tadenuma, 1996), the earlier studies in the literature considered the case where each agent can be assigned at most one object (*single-object-demand model*). This problem has been extensively studied from either the viewpoint of mechanism design (Schummer, 2000; Miyagawa, 2001; Svensson and Larsson, 2002) or from the normative viewpoint (Svensson, 1983; Alkan et al., 1991; Tadenuma and Thomson, 1991). The latter studies focused on efficiency and fairness axioms, especially *no-envy* (also known as *envy-freeness:* no agent prefers another agent's bundle to her own) and *egalitarian-equivalence* (each agent should be indifferent between her bundle and a common 'reference' bundle). Note that when agents have single-object demands, by Svensson (1983), *no-envy* implies *assignment-efficiency* (in each economy, objects are allocated in order to maximize the total valuation of the agents or to minimize the total cost incurred by the agents). The studies from the strategic viewpoint generally focused on *strategy-proofness* (reporting the true preferences is a weakly dominant strategy for all agents) which is the incentive compatibility criteria used when neither any agent nor the center knows other agents' preferences or the likelihood of others' preferences. These studies characterized mechanisms that satisfy *strategy-proofness, budget-balance* (total monetary transfer adds up to a certain fixed amount in all economies) and auxiliary axioms, but none of these mechanisms are *envy-free.*

The papers from the normative viewpoint have not assumed private information. An exception is Tadenuma and Thomson (1995) who showed, for the case of allocating a single indivisible good and money, that there is no *strategy-proof, envy-free,* and *budget-balanced* mechanism. As in most economic environments, by Green and Laffont (1977), *strategy-proofness*, *assignment-efficiency,* and *budget balance* are generally incompatible in the problem of allocating objects and money (the queueing problem is an exception). Hence, one has to give up either *strategy-proofness* or *budget-balance* in order to obtain *assignment-efficient* and fair mechanisms. In recent years, a new strand of literature, which adopts both the normative and strategic viewpoint and studies the joint implications of *strategy-proofness* and fairness axioms, has developed. By Holmström (1979), which applies in our context, a mechanism is *assignment-efficient* and *strategy-proof* if and only if it is a Groves mechanism. Imposing an additional fairness criteria like *egalitarian equivalence* or *no-envy* gives us a subset of the class of Groves mechanisms.

In general, papers in this strand of literature (Ashlagi and Serizawa, 2012; Atlamaz and Yengin, 2008; Chun, 2006; Chun, Mitra, and Mutuswami, 2014; Hashimoto and Saitoh, 2012; Ohseto, 2004, 2006; Mukherjee, 2014) are concerned with the case where agents have single-object demands. In only few papers (Pápai, 2003; Yengin, 2012a,b, 2013a,b), the more complex problem of allocating heterogeneous objects when each agent can be assigned any number of objects (*multi-object-demand model*) is studied. It is to this strand of literature on the Groves mechanisms that our paper belongs to. We combine both normative and strategic approaches and investigate mechanisms that satisfy three criteria: *assignment-efficiency*, *strategy-proofness*, and fairness.

[3] Examples include choosing the locations of desirable facilities or events (state capitals, parks, international airports, etc.), auctions held to allocate water entitlements to farmers; the allocation of fishing or pollution permits, allocation of community housing or charitable goods and money among the needy, managing the use of commonly owned indivisible goods in cooperative enterprises such as cooperative supported agriculture, allocation of inheritance among heirs.

Ohseto (2006) characterizes *envy-free* Groves mechanisms in the single-object-demand model of allocating homogenous objects[4] and money. In the multi-object-demand model, Pápai (2003) characterizes the *envy-free* Groves mechanisms on the domain of subadditive cost functions (superadditive valuation functions). *Envy-free* Groves mechanisms are also characterized in the queueing problem (Chun, 2006; Kayi and Ramaekers, 2010). *Egalitarian-equivalent* Groves mechanisms have been characterized by Ohseto (2004) in the single-object-demand model; by Chun, Mitra, and Mutuswami (2014) in the queueing problem; and by Yengin (2012b) in the multi-object-demand model.

No-envy and *egalitarian-equivalence,* are arguably, the two central concepts proposed to evaluate the fairness of an allocation in the axiomatic fair allocation literature and have been studied in several different economic models.[5] These two axioms are motivated by different ethical considerations and a reconciliation would be desirable. In this paper, we investigate these two axioms in a model where their joint implications have not been studied so far, namely the multi-object-demand model with private information. First, in Theorem 1, we present a complete characterization of the class of mechanisms satisfying *assignment-efficiency, no-envy, egalitarian equivalence*, and *strategy-proofness* on the domain of subadditive cost functions (we show that the first three axioms are incompatible when cost functions are superadditive or unrestricted). When there are at least three agents (Corollary 2), or under *population monotonicity* (no agent gets worse off when population increases, Corollary 3), or under an upper bound $T \in \mathbb{R}$ on the deficits generated by the mechanisms ($T-$*bounded-deficit,* Corollary 4), the characterized class turns out to be *welfare-egalitarian* (all agents enjoy the same utility level). Second, under $T-$*bounded-deficit*, we identify the Pareto-dominant subclass. We show that, under $T-$*bounded-deficit,* the mechanisms in this subclass are not Pareto-dominated by any other Groves mechanism and they actually Pareto-dominate other subclasses of Groves mechanisms satisfying different fairness axioms (Theorem 3).

There is an extensive literature studying egalitarian solutions (see Ginés and Marhuenda, 2000, and references therein). Most of the characterizations of *welfare-egalitarianism* rely on monotonicity and solidarity axioms (see, for instance, our companion paper, Yengin, 2012a). Unlike the previous literature, here, the axioms we consider to characterize the *welfare-egalitarian* solutions are not solidarity axioms, but the two most fundamental equity concepts in the fair division literature, namely, *no-envy* and *egalitarian-equivalence*. No similar characterization of *welfare-egalitarianism* based on these two axioms exists in other models.

The results we present here are the only ones so far on the compatibility of *no-envy* and *egalitarian-equivalence* in the multi-object-demand setting. Actually, to our knowledge, ours is the only result in the literature showing the compatibility of these two axioms. It has been shown that in several economic problems, these two axioms are incompatible. As showed by Postlewaite (quoted by Daniel, 1978) there are well-behaved exchange economies where all *egalitarian-equivalent* and Pareto-efficient allocations violate *no-envy*. In time division problems (division of a one-dimensional, non-homogeneous, and atomless continuum, when each agent is to receive an interval), no *egalitarian-equivalent* mechanism is *envy-free* (Thomson, 1996). The same incompatibility also exists in the literature on indivisible goods and money allocation. In the single-object-demand model where heterogeneous objects and money are allocated, Thomson (1990) showed that under *budget-balance*, no mechanism is *egalitarian-equivalent* and *envy-free*. Also, by Chun, Mitra, and Mutuswami (2014), when there are at least 4 agents, *no-envy* and *egalitarian-equivalence* are incompatible in the queueing problem (assignment of agents to queue positions when monetary transfers are possible). A weaker result has been proven by Chun (2006) who showed that the two equity notions are incompatible in the queueing problem even under *budget-balance*. Note

[4] The value of each object is the same for a given agent.

[5] For a survey, see Thomson (2011).

that considering queue positions as objects, a queueing problem is a special case of the problem of allocating heterogeneous objects when each agent gets exactly one object.

The contrast between the compatibility result we obtain here and the impossibility results obtained in the previous literature on the allocation of objects and money is striking. Our results can not simply build on the known relationships of *no-envy* with *assignment-efficiency* or with *egalitarian-equivalence* in the previous literature since most of those relationships no longer hold in our model. For instance, in the single-object-demand model, *no-envy* implies *assignment-efficiency* which is not the case in our model (with or without imposing *budget-balance*, see Example 1). Hence, if we want to assign the objects efficiently, simply imposing *no-envy* will not be enough, we need to impose *assignment-efficiency* as a separate axiom.

A few aspects differentiate our model from the previous papers in the literature. First, we allow each agent to be assigned more than one object. Hence, the structure of the cost functions (i.e., whether they are superadditive, subadditive, etc.) is also important. Second, we assume that agents' preferences/costs are their private information (we impose *strategy-proofness* and forgo *budget-balance*). Third, we allow the empty set to be a reference set of objects when choosing *egalitarian-equivalent* allocations. It is natural to ask which of these three aspects underlie the disparity between our compatibility result and the related results in the literature. Figuring out the underlying reasons is important for our understanding of how objects and money should be distributed in different settings.

In Subsection 3.1, first, we show that when cost functions are *unrestricted* or *superadditive, assignment-efficiency, no-envy,* and *egalitarian-equivalence* are incompatible (Proposition 1a). Then, in Proposition 1b,c,d we present a positive result: if cost functions are *additive* or *subadditive*, or when the mechanism is defined over problems which only consist of two agents, then *assignment-efficiency, no-envy,* and *egalitarian-equivalence* are compatible in the multi-object-demand model. We do not need to impose *strategy-proofness* to obtain this compatibility result. This result still holds even if we additionally require *budget-balance.* In other words, *Pareto-efficiency* (*assignment-efficiency* together with *budget-balance*), *no-envy,* and *egalitarian-equivalence* are compatible in our model. Proposition 1d demonstrates that the compatibility result does not require the empty set to be a reference set in each economy when choosing *egalitarian-equivalent* allocations. On the other hand, we need to allow the empty set to be a reference set in some economies in order to obtain this compatibility (Proposition 2). Altogether, our results show that we need three factors to simultaneously exist in order to achieve the compatibility of *no-envy* and *egalitarian-equivalence:* agents can be assigned any number of objects, cost functions are *additive* or *subadditive*[6], and the empty set can be a reference set in *egalitarian-equivalent* allocations. Hence, the other factors, namely, imposing *strategy-proofness* and forgoing *budget-balance*, are not playing a role in obtaining this compatibility, although the compatibility is retained when we separately introduce these factors as well. To sum up, our paper provides a significant addition to our understanding of the relationships between *no-envy* and *egalitarian-equivalence* within the literature on allocation of objects and money and shows how these relationships depend on different specifications of the model.

In conclusion, our contribution is threefold. First, we contribute to the literature on fair Groves mechanisms by characterizing Groves mechanisms satisfying *no-envy* and *egalitarian-equivalence* in a multi-object-demand setting where the joint implication of these two axioms has not yet been investigated. Secondly, we provide an alternative foundation for *welfare-egalitarianism* based on *no-envy* and *egalitarian-equivalence.* Thirdly, in the problem of allocating objects and money, we clarify how changes in the model specifications affect the results on the compatibility of *no-envy* and *egalitarian-equivalence.*

[6] If we are allocating indivisible goods, then valuation functions need to be *additive* or *superadditive.*

2 Preliminaries

2.1 The Model

A finite set of heterogeneous indivisible tasks is to be allocated among a finite set of agents by a "center". All tasks must be allocated. An agent can be assigned either no task, a single task, or more than one task. Each task is assigned to only one agent. Let $\mathbb{A}$ be the finite set of tasks, with $|\mathbb{A}| \geq 1$, and α, β be typical elements of $\mathbb{A}$.

There is an infinite set of "potential" agents indexed by the positive natural numbers $\mathbb{N} \equiv \{1, 2, ...\}$. In any given problem, only a finite number of them are present. Let $\mathcal{N}$ be the set of finite subsets of potential agents with at least two agents. Let N with $|N| = n \geq 2$ be a typical element of $\mathcal{N}$. The number of agents may be smaller than, equal to, or greater than the number of tasks.

Let $2^{\mathbb{A}}$ be the set of subsets of $\mathbb{A}$. Each agent i has a cost function $c_i : 2^{\mathbb{A}} \rightarrow \mathbb{R}_+$ with $c_i(\emptyset) = 0$.[7] We refer to such a cost function as *unrestricted.* Let $\mathcal{C}_{un}$ be the set of all such functions.

If for each $A \in (2^{\mathbb{A}} \backslash \emptyset)$, $c_i(A) = \sum\limits_{\alpha \in A} c_i(\{\alpha\})$, then c_i is *additive.* If for each pair $\{A, A'\} \subseteq 2^{\mathbb{A}}$ with $A \cap A' = \emptyset$, $c_i(A \cup A') \leq c_i(A) + c_i(A')$, then c_i is *subadditive*, and if for each $\{A, A'\} \subseteq 2^{\mathbb{A}}$ with $A \cap A' = \emptyset$, $c_i(A \cup A') \geq c_i(A) + c_i(A')$, then c_i is *superadditive.* Let $\mathcal{C}_{ad}, \mathcal{C}_{sub}$, and $\mathcal{C}_{sup}$ be the classes of additive, subadditive, and superadditive cost functions, respectively. Let $\mathcal{C}$ be a generic element of $\{\mathcal{C}_{un}, \mathcal{C}_{ad}, \mathcal{C}_{sub}, \mathcal{C}_{sup}\}$ and $\mathcal{C}^N$ be the $n-$fold Cartesian product of $\mathcal{C}$. In the rest of the paper, the results hold on any domain unless otherwise stated.

For each $N \in \mathcal{N}$, a *cost profile for* N is a list $c \equiv (c_1, ..., c_n)$. Let $\bigcup\limits_{N \in \mathcal{N}} \mathcal{C}^N$ be the domain of cost profiles where for each $i \in \mathbb{N}$, $c_i \in \mathcal{C}$.

A cost profile defines an *economy.* Let $c, c', \widehat{c}$ be typical economies with associated agent sets $N, N', \widehat{N}$. For each $N \in \mathcal{N}$ and each $i \in N$, let c_{-i} be the cost profile of the agents in $N \backslash \{i\}$. For each $N \in \mathcal{N}$, each $N' \subseteq N$, and each $c \in \mathcal{C}^N$, let $c_{N'}$ be the restriction of c to $N' : c_{N'} \equiv (c_i)_{i \in N'}$.

There is a perfectly divisible good we call "money". Let t_i denote agent i's consumption of the good. We call t_i agent *i's transfer*: if $t_i > 0$, it is a transfer from the center to i; if $t_i < 0$, $|t_i|$ is a transfer from i to the center.

The center assigns the tasks and determines each agent's transfer. Agent i's utility when she is assigned the set of tasks $A_i \in 2^{\mathbb{A}}$ (note that A_i may be empty) and consumes $t_i \in \mathbb{R}$ is

$$u(A_i, t_i; c_i) = -c_i(A_i) + t_i. \tag{1}$$

An *assignment for* N is a list $(A_i)_{i \in N}$ such that $\bigcup\limits_{i \in N} A_i = \mathbb{A}$ and for each pair $\{i, j\} \subseteq N$, $A_i \cap A_j = \emptyset$. For each $N \in \mathcal{N}$, let $\mathcal{A}(N)$ be the set of all possible assignments for N.

A *transfer profile for* N is a list $(t_i)_{i \in N} \in \mathbb{R}^N$. An *allocation for* N is a list $(A_i, t_i)_{i \in N}$ where $(A_i)_{i \in N}$ is an assignment and $(t_i)_{i \in N}$ is a transfer profile for N.

A *mechanism* is a function $\varphi \equiv (A, t)$ defined over the union $\bigcup\limits_{N \in \mathcal{N}} \mathcal{C}^N$ that associates with each economy an allocation: for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$, $\varphi_i(c) \equiv (A_i(c), t_i(c)) \in 2^{\mathbb{A}} \times \mathbb{R}$.

For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, let $W(c)$ be the minimal total cost among all possible assignments for N. That is,

$$W(c) = \min \left\{ \sum_{i \in N} c_i(A'_i) : (A'_i)_{i \in N} \in \mathcal{A}(N) \right\}.$$

[7] As usual, $\mathbb{R}_+$ denotes the set of non-negative real numbers.

2.2 The Mechanisms

Since there is no restriction on the size of individual or total transfer, every allocation is Pareto-dominated by another allocation with higher transfers. On the other hand, since utilities are quasi-linear, given a cost profile c, an allocation that minimizes the total cost is Pareto-efficient for c among all allocations with the same, or smaller, total transfer. Our first axiom requires mechanisms to choose only such allocations.

Assignment-Efficiency: For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum_{i \in N} c_i(A_i(c)) = W(c)$.

We work with single-valued mechanisms and assume that each *assignment-efficient* mechanism is associated with a tie-breaking rule τ that determines, for each economy, which of the efficient assignments (if there are more than one) is chosen by the mechanism. Let $\mathcal{T}$ be the set of all possible tie-breaking rules.

Since costs are private information, an *assignment-efficient* mechanism assigns the tasks so that the actual total cost is minimal only if agents report their true costs. Similarly, truthful revelation of costs is essential in order to determine fair allocations. Then, a desirable property for a mechanism is that no agent should ever benefit by misrepresenting her costs (Gibbard, 1973; Satterthwaite, 1975).

Strategy-Proofness: For each $N \in \mathcal{N}$, each $i \in N$, each $c \in \mathcal{C}^N$, and each $c'_i \in \mathcal{C}$, $u(\varphi_i(c); c_i) \geq u(\varphi_i(c'_i, c_{-i}); c_i)$.

Next, we introduce the class of mechanisms that will be our focus: *Egalitarian mechanisms.*

Let $\gamma : \mathcal{N} \to \mathbb{R}$ be an arbitrary function that associates each population with a real number and Γ be the set of all such functions.

The *Egalitarian* mechanism associated with $\gamma \in \Gamma$ and $\tau \in \mathcal{T}$, $E^{\gamma,\tau}$:
Let $E^{\gamma,\tau} \equiv (A^\tau, t^{\gamma,\tau})$ be such that for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $(A^\tau_i(c))_{i \in N}$ is an efficient-assignment for c and for each $i \in N$,

$$t_i^{\gamma,\tau}(c) = -\sum_{j \in N \setminus \{i\}} c_j(A_j^\tau(c)) + \gamma(N). \tag{2}$$

The transfers of an *Egalitarian mechanism* have a simple structure: each agent pays the sum of the costs incurred by the other agents at the efficient assignment chosen by the mechanism and receives a sum of money $\gamma(N) \in \mathbb{R}$ that may depend on the population, but is independent of the cost profile in the economy. That is, in all economies with the same agent set N, the amount that each agent receives is $\gamma(N) \in \mathbb{R}$. Different choices for γ correspond to different selections from the class of *Egalitarian mechanisms*.

Let $\mathcal{E}^\gamma \equiv \{E^{\gamma,\tau} | \tau \in \mathcal{T}\}$. Note that for each $\gamma \in \Gamma$, the mechanisms in $\mathcal{E}^\gamma$ are *Pareto-indifferent*.[8] That is, the particular tie-breaking rule used is irrelevant in the determination of the utilities. Let $\mathcal{E} \equiv \bigcup_{\gamma \in \Gamma} \mathcal{E}^\gamma$ be the class of the *Egalitarian mechanisms*. The mechanisms in this class equalize welfare of all agents:

Welfare-Egalitarianism: For each $N \in \mathcal{N}$, each pair $\{i, j\} \subseteq N$, and each $c \in \mathcal{C}^N$, $u(\varphi_i(c); c_i) = u(\varphi_j(c); c_j)$.

The proof of the following result is in Yengin (2012a):

[8] Let $N \in \mathcal{N}$ and $c \in \mathcal{C}^N$. Allocations $(A_i, t_i)_{i \in N}$ and $(A'_i, t'_i)_{i \in N}$ are *Pareto-indifferent* for c if and only if for each $i \in N$, $u(A_i, t_i; c_i) = u(A'_i, t'_i; c_i)$. The mechanisms φ and φ' are *Pareto-indifferent* if for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $u(\varphi_i(c); c_i) = u(\varphi'_i(c); c_i)$.

Lemma 1. *A mechanism is assignment-efficient, strategy-proof, and welfare-egalitarian on* $\bigcup_{N\in\mathcal{N}} \mathcal{C}^N$ *if and only if it belongs to* $\mathcal{E}$.

The class of *Egalitarian* mechanisms is a subclass of the well-known class of Groves mechanisms (Vickrey, 1961; Clarke, 1971; and Groves, 1973). For each $i \in \mathbb{N}$, let h_i be a real-valued function defined over the union $\bigcup_{N\in\mathcal{N}} \mathcal{C}^N$ such that for each $N \in \mathcal{N}$ with $i \in N$ and each $c \in \mathcal{C}^N$, h_i depends only on c_{-i}. Let $h = (h_i)_{i\in\mathbb{N}}$ and $\mathcal{H}$ be the set of all such h.

The *Groves* mechanism associated with $h \in \mathcal{H}$ and $\tau \in \mathcal{T}$, $G^{h,\tau}$:
Let $G^{h,\tau} \equiv (A^\tau, t^{h,\tau})$ *be such that for each* $N \in \mathcal{N}$ *and each* $c \in \mathcal{C}^N$, $(A_i^\tau(c))_{i\in N}$ *is an efficient-assignment for* c *and for each* $i \in N$,

$$t_i^{h,\tau}(c) = -\sum_{j\in N\setminus\{i\}} c_j(A_j^\tau(c)) + h_i(c_{-i}) = -W(c) + c_i(A_i^\tau(c)) + h_i(c_{-i}). \tag{3}$$

The transfer of each agent determined by a Groves mechanism (i.e., Groves transfer) has two parts. First, each agent pays the total cost incurred by all other agents at the assignment chosen by the mechanism. Second, each agent i receives a sum of money $h_i(c_{-i}) \in \mathbb{R}$ that does not depend on her own cost c_i.

We have the following equation which will be of much use.[9] For each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$,

$$u(G_i^{h,\tau}(c); c_i) = -W(c) + h_i(c_{-i}). \tag{4}$$

The following result follows from Holmström (1979) since for each $N \in \mathcal{N}$, $\mathcal{C}^N$ is convex:

Lemma 2. *A mechanism is assignment-efficient and strategy-proof on* $\bigcup_{N\in\mathcal{N}} \mathcal{C}^N$ *if and only if it is a Groves mechanism.*

By Lemmas 1 and 2, the class $\mathcal{E}$ is a subclass of the class of Groves mechanisms. Note that $E^{\gamma,\tau} = G^{h,\tau}$, where for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}^N$, $h_i(c_{-i}) = \gamma(N)$.

A mechanism satisfies *budget-balance* if there is $T \in \mathbb{R}$ such that for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum_{i\in N} t_i(c) = T$. A mechanism is *Pareto-efficient* if it is *assignment-efficient* and *budget-balanced.* It follows from Green and Laffont (1977) that *Pareto-efficiency* and *strategy-proofness* are not compatible in our model. Hence, the obvious limitation of the Groves mechanisms is that they do not balance the budget. In other words, if we require truthful revelation of the preferences/costs in order to realize an efficient and fair allocation with respect to the true costs, then we need to pay the price in terms of budget imbalances. This drawback of the Groves mechanisms is mitigated by the fact that there are Groves mechanisms that generate deficits bounded above. That is, deficits never exceed a given amount $T \in \mathbb{R}$ in any economy, no matter what the cost functions of the agents are.

Let $T \in \mathbb{R}$.

$T-$**Bounded-Deficit:** For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum_{i\in N} t_i(c) \leq T$.

If a mechanism satisfies $T-$*bounded-deficit*, then for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, it generates a budget surplus (earnings for the center) $-\sum_{i\in N} t_i(c)$ which is at least $-T$. Hence, the surplus is bounded below by $-T$. If a mechanism satisfies $T-$*bounded-deficit* where $T = 0$, then the center never incurs a deficit in any economy (*no-deficit*).

[9] By (4), for each $h \in \mathcal{H}$, the mechanisms in $\{G^{h,\tau}\}_{\tau\in\mathcal{T}}$ are *Pareto-indifferent*.

2.3 Fairness Axioms

Perhaps, the most well-known notion of fairness is *no-envy* (Foley 1967), which requires that each agent should find her bundle at least as desirable as any other agent's bundle. Hence, given the opportunity of choosing among all the bundles compromising an allocation, an agent should choose her assigned bundle.

No-Envy: For each $N \in \mathcal{N}$, each pair $\{i,j\} \subseteq N$, and each $c \in \mathcal{C}^N$,

$$u(\varphi_i(c); c_i) \geq u(\varphi_j(c); c_i).$$

An alternative central notion of fairness, *egalitarian-equivalence* (Pazner and Schmeidler, 1978), requires that only those allocations such that each agent is indifferent between her assigned bundle and a common *reference bundle* (consisting of a *reference set of tasks* and a *reference transfer*) should be chosen.

Egalitarian-Equivalence: For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, there are a reference set of tasks (which may be empty) $R(c) \in 2^{\mathbb{A}}$ and a reference transfer $r(c) \in \mathbb{R}$ such that for each $i \in N$,

$$u(\varphi_i(c); c_i) = u((R(c), r(c)); c_i).$$

In Subsection 3.3, we also consider the following fairness notions:

Unanimity is a weak fairness axiom which requires that if everyone has the same cost function, then all of them should have the same welfare.

Unanimity: For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$ such that $c_i = c_j$ for each $\{i,j\} \subseteq N$,

$$u(\varphi_i(c); c_i) = u(\varphi_j(c); c_j) \text{ for each } \{i,j\} \subseteq N.$$

If $c_i(A) \geq c_j(A)$ for each $A \in 2^{\mathbb{A}}$, we write $c_i \geq c_j$. Suppose that $c_i \geq c_j$. If j were assigned a lower utility than i, it would be as if j were penalized for having lower costs. The following property is meant to prevent this situation.

Order Preservation: For each $N \in \mathcal{N}$, each pair $\{i,j\} \subseteq N$, and each $c \in \mathcal{C}^N$ such that $c_i \geq c_j$,

$$u(\varphi_i(c); c_i) \leq u(\varphi_j(c); c_j).$$

Note that *order preservation* implies *unanimity*. By Proposition 3 in Yengin (2012b), if a Groves mechanism is *egalitarian-equivalent* or *envy-free*, then it *preserves order*.

Suppose some of the agents experience an increase in the cost of performing some tasks. The cost of an efficient assignment in this new cost profile is at least as much as the original one. One can also interpret the increase in the cost of an agent to perform some tasks as a decrease in the skill of that agent to perform those tasks. If one holds the Rawlsian view that agents' skills are a common asset for the society, then a deterioration in the skill profile of the society is bad news for everyone. Accordingly, *solidarity* calls for that no agent should be better off after this bad news.

Solidarity: For each $N \in \mathcal{N}$, each $N' \subseteq N$, each pair $\{c, c'\} \subset C^N$ such that for each $j \in N'$, $c'_j \geq c_j$ and $c'_{N \setminus N'} = c_{N \setminus N'}$, and each $i \in N$,

$$u(\varphi_i(c); c_i) \geq u(\varphi_i(c'); c'_i).$$

3 The Results

No-envy and *egalitarian-equivalence* have been investigated in numerous economic models; however, their compatibility is hard to achieve. We compare our results on the compatibility with the results obtained in the *single-object-demand model* where each agent is assigned at most one object and monetary transfers are allowed. This model includes the case where there is only one object to be allocated as well as the cases where a finite set of homogenous objects (for each agent, the value of each of the objects is the same) or a finite set of heterogeneous objects (in this case, it is assumed that $|N| = |\mathbb{A}|$ and each agent is assigned exactly one object) are to be allocated. The queueing problem belongs to the last case.

In the *classical single-object-demand model* considered by the earlier papers in the literature on the allocation of objects and money, agents' preferences are not their private information and *budget-balance* is always required. Thomson (2011) summarizes the following results obtained in this classical single-object-demand model:

∗ If a single object is to be allocated, then the *assignment-efficient* allocation where the winner of the object is indifferent between her bundle and the losers' common bundle is *envy-free* and *egalitarian-equivalent* (here, the reference set is the empty set).

∗ If there are only two agents and two heterogeneous objects in the economy (i.e., $|N| = |\mathbb{A}| = 2$), any *assignment-efficient* and *egalitarian-equivalent* mechanism is also *envy-free.*

∗ If heterogeneous objects are to be allocated and a mechanism is defined for economies including those such that $|N| = |\mathbb{A}| \geq 3$, then, by Thomson (1990), there is no *assignment-efficient*, *egalitarian-equivalent*, and *envy-free* mechanism. Note that in Thomson (1990), each reference set used in determining *egalitarian-equivalent* allocations contains a single object, i.e., the reference sets can not be empty or contain multiple objects.

The single-object-demand model has also been studied in the recent literature on fairness of Groves mechanisms; however, the compatibility of *no-envy* and *egalitarian-equivalence* has only been investigated in the special case of this model, namely, the queueing problem. Chun, Mitra, and Mutuswami (2014) showed that in the queueing problem, if there are at least 4 agents, then *no-envy* and *egalitarian-equivalence* are incompatible; whereas if there are at most 3 agents in the problem, then *no-envy, egalitarian-equivalence, strategy-proofness* are compatible.

We consider the *multi-object-demand model* for allocating heterogeneous objects where monetary transfers are allowed. In our setting, we assume the following:

(i) each agent can be assigned any number of objects (hence, the number of agents may be different than the number of objects to be allocated),

(ii) cost functions can be *unrestricted, additive, subadditive,* or *superadditive,*

(iii) empty set is allowed to be a reference set when choosing *egalitarian-equivalent* allocations,

(iv) the cost functions of agents are their private information and we impose *strategy-proofness* as our incentive compatibility condition.

The compatibility of *no-envy* and *egalitarian-equivalence* has not yet been studied in the multi-object-demand model with or without imposing *strategy-proofness.* We investigate whether and to what extend the aspects which differentiate our setting from the single-object-demand model lead to differences in the results on the compatibility of *no-envy* and *egalitarian-equivalence.* First, in Subsection 3.1, we introduce points (i), (ii), and (iii) and compare the results obtained in the single-object-demand model with our results in the multi-object-demand case. Then, in Subsection 3.2, we introduce the other feature of our model, point (iv), and investigate how the compatibility results in the multi-object-demand model change under *strategy-proofness.*

3.1 Compatibility Under Multi-Object-Demand

In this section, we investigate the relationship between *no-envy* and *egalitarian-equivalence* in the multi-object-demand model and compare our results with the known relationships in the single-object-demand model. Note that in the single-object-demand model, *no-envy* implies *assignment-efficiency* (Svensson, 1983). This is not the case in the multi-object-demand model as the following example demonstrates (for an example under *budget-balance*, see Proposition 2.1 in Beviá, 1998).

Example 1. Let $N = \{1,2\}$ and $\mathbb{A} = \{\alpha, \beta, \theta, \phi\}$. Let $c_1^* \in \mathcal{C}_{ad}$ be such that $c_1^*(\{\alpha\}) = 4$, $c_1^*(\{\beta\}) = 8$, $c_1^*(\{\theta\}) = 2$, and $c_1^*(\{\phi\}) = 14$. Let $c_2^* \in \mathcal{C}_{ad}$ be such that $c_2^*(\{\alpha\}) = 5$, $c_2^*(\{\beta\}) = 7$, $c_2^*(\{\theta\}) = 5$, and $c_2^*(\{\phi\}) = 3$. Let $\gamma \in \Gamma$ and $\varphi^\tau \equiv (A^\tau, t^\tau)$ be such that for each $c \in \bigcup_{N \in \mathcal{N}} \mathcal{C}^N \backslash \{c^*\}$, $\varphi^\tau(c) = E^{\gamma,\tau}(c)$; and $A_1^\tau(c^*) = \{\alpha, \beta\}$, $A_2^\tau(c^*) = \{\theta, \phi\}$, and $(t_1^\tau(c^*), t_2^\tau(c^*)) = (2, 1)$. Note that for each $c \in \bigcup_{N \in \mathcal{N}} \mathcal{C}^N \backslash \{c^*\}$, $\varphi^\tau(c)$ is an *envy-free* and *assignment-efficient* allocation for c. Since $-c_1^*(A_1^\tau(c^*)) + t_1^\tau(c^*) > -c_1^*(A_2^\tau) + t_2^\tau(c^*)$ and $-c_2^*(A_2^\tau) + t_2^\tau(c^*) > -c_2^*(A_1^\tau) + t_1^\tau(c^*)$, $\varphi^\tau(c^*)$ is an *envy-free* allocation for c^*. Note that $c_1^*(A_1^\tau(c^*)) + c_2^*(A_2^\tau(c^*)) > c_1^*(\{\alpha, \theta\}) + c_2^*(\{\beta, \phi\})$. Hence, $A^\tau(c^*)$ is not an *efficient assignment* for c^*. That is, φ^τ is *envy-free* but not *assignment-efficient*.

In the problem of allocating heterogeneous objects and money, does the incompatibility of *no-envy* and *egalitarian-equivalence* that exists in the single-object-demand model disappear in the multi-object-demand case? Proposition 1a shows that the answer is negative on the *superadditive* or on the *unrestricted domain*. On the other hand, on the *additive* or the *subadditive* domain, Proposition 1 (c) and (d) show that, *assignment-efficiency*, *egalitarian-equivalence*, and *no-envy* are compatible. On the domain of economies with two agents, the compatibility of *assignment-efficiency*, *egalitarian-equivalence*, and *no-envy* obtained in the classical single-object-demand model extends to our model for any domain of cost profiles (Proposition 1b). The compatibility results obtained in Proposition 1 are retained under *budget-balance*.

Proposition 1. ***a)*** *On the unrestricted or the superadditive domain, there is no assignment-efficient, egalitarian-equivalent, and envy-free mechanism.*

b) *On the domain of two-agent economies,* $\bigcup_{N \in \mathcal{N}:|N|=2} \mathcal{C}^N$ *where* $\mathcal{C} \in \{\mathcal{C}_{un}, \mathcal{C}_{ad}, \mathcal{C}_{sub}, \mathcal{C}_{\sup}\}$, *there are assignment-efficient, egalitarian-equivalent, and envy-free mechanisms.*

c) *On the additive or the subadditive domain or when* $|\mathbb{A}| = 1$, *if an assignment-efficient mechanism is egalitarian-equivalent such that for each economy, the reference set of tasks is the empty set (i.e., it is welfare-egalitarian), then it is envy-free.*

d) *On the additive or the subadditive domain, there are assignment-efficient, egalitarian-equivalent, and envy-free mechanisms which are not welfare-egalitarian.*

For the proof of Proposition 1, we will use the following Lemma. Note that in the multi-object-demand model, by Proposition 2.2 in Beviá (1998), *envy-free* mechanisms exist; and it is easy to construct *egalitarian-equivalent* mechanisms by Lemma 3a,b below.

Lemma 3. ***a)*** *A mechanism* $\varphi \equiv (A, t)$ *is egalitarian-equivalent if and only if for each* $N \in \mathcal{N}$ *and each* $c \in \mathcal{C}^N$, *there are a reference set of tasks (which may be empty)* $R(c) \in 2^{\mathbb{A}}$ *and a reference transfer* $r(c) \in \mathbb{R}$ *such that for each* $i \in N$,

$$t_i(c) = c_i(A_i(c)) - c_i(R(c)) + r(c). \tag{5}$$

b) *A mechanism* $\varphi^\tau \equiv (A^\tau, t^\tau)$ *is Pareto-efficient (i.e., assignment-efficient and budget-balanced) and egalitarian-equivalent if and only if there is* $T \in \mathbb{R}$ *and for each* $N \in \mathcal{N}$ *and each* $c \in \mathcal{C}^N$, *there* *are* $R(c) \in 2^{\mathbb{A}}$ *and* $r(c) \in \mathbb{R}$ *such that for each* $i \in N$,

$$t_i^{\tau}(c) = c_i(A_i^{\tau}(c)) - c_i(R(c)) + \frac{1}{n}[T - W(c) + \sum_{i \in N} c_i(R(c))]. \tag{6}$$

c) *A mechanism* $\varphi \equiv (A, t)$ *is egalitarian-equivalent and envy-free if and only if for each* $N \in \mathcal{N}$ *and each* $c \in \mathcal{C}^N$*, there are* $R(c) \in 2^{\mathbb{A}}$ *and* $r(c) \in \mathbb{R}$ *such that (5) holds and for each pair* $\{i, j\} \subseteq N$,

$$c_i(A_j(c)) - c_j(A_j(c)) \geq c_i(R(c)) - c_j(R(c)). \tag{7}$$

d) *A mechanism* $\varphi \equiv (A, t)$ *is welfare-egalitarian if and only if it is egalitarian-equivalent such that for each economy, the reference set of tasks is the empty set.*

Proof of Lemma 3:

a) Follows from (1) and the definition of *egalitarian-equivalence*.

b) Let $\varphi^{\tau} \equiv (A^{\tau}, t^{\tau})$ be *Pareto-efficient* and *egalitarian-equivalent*. By (5) and *budget-balance*, there is $T \in \mathbb{R}$ and for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, there are $R(c) \in 2^{\mathbb{A}}$ and $r(c) \in \mathbb{R}$ such that $\sum_{i \in N} t_i^{\tau}(c) = \sum_{i \in N} c_i(A_i^{\tau}(c)) - \sum_{i \in N} c_i(R(c)) + nr(c) = T$. That is, $r(c) = \frac{1}{n}[T - W(c) + \sum_{i \in N} c_i(R(c))]$. This equality and (5) together imply (6). Conversely, let $\varphi^{\tau} \equiv (A^{\tau}, t^{\tau})$ be an *assignment-efficient* mechanism such that for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$, (6) holds. By (5), φ^{τ} is *egalitarian-equivalent* where for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $r(c) = \frac{1}{n}[T - W(c) + \sum_{i \in N} c_i(R(c))]$. Since for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum_{i \in N} t_i^{\tau}(c) = T$, φ^{τ} is *budget-balanced*.

c) Let $\varphi \equiv (A, t)$ be *egalitarian-equivalent* and *envy-free*. By *egalitarian-equivalence*, for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, there are $R(c) \in 2^{\mathbb{A}}$ and $r(c) \in \mathbb{R}$ such that (5) holds. By *no-envy*, (1) and (5), for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each pair $\{i, j\} \subseteq N$, $-c_i(R(c)) + r(c) \geq -c_i(A_j(c)) + c_j(A_j(c)) - c_j(R(c)) + r(c)$. That is, (7) holds. Conversely, let $\varphi \equiv (A, t)$ be such that for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, there are $R(c) \in 2^{\mathbb{A}}$ and $r(c) \in \mathbb{R}$ such that for each pair $\{i, j\} \subseteq N$, (5) and (7) holds. By part (a), $\varphi \equiv (A, t)$ is *egalitarian-equivalent*. By (5), for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $\{i, j\} \subseteq N$, $c_i(R(c)) - c_j(R(c)) = c_i(A_i(c)) - c_j(A_j(c)) - t_i(c) + t_j(c)$. This equality and (7) together imply that for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $\{i, j\} \subseteq N$, $-c_i(A_i(c)) + t_i(c) \geq -c_i(A_j(c)) + t_j(c)$. By (1), φ is *envy-free*.

d) Since utilities are quasilinear, follows from (5). □

Proof of Proposition 1:

a) Let $\mathcal{C} \in \{\mathcal{C}_{un}, \mathcal{C}_{sup}\}$. Assume, by contradiction, that on $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$, $\varphi^{\tau} \equiv (A^{\tau}, t^{\tau})$ is *assignment-efficient*, *envy-free*, and *egalitarian-equivalent*. Let $N = \{1, 2, 3\}$, $\mathbb{A} = \{\alpha, \beta, \theta\}$, and $c \in \mathcal{C}^N$ be such that

(i) $c_1(\{\alpha\}) = c_1(\{\beta\}) = c_3(\{\alpha\}) = c_3(\{\theta\}) = 2$,

(ii) $c_1(\{\theta\}) = c_2(\{\beta\}) = c_2(\{\theta\}) = c_3(\{\beta\}) = 3$,

(iii) $c_2(\{\alpha\}) = 4$,

(iv) for each pair $\{i, j\} \subset N$ and each $A \subseteq \mathbb{A}$ with $|A| \geq 2$, $c_i(A) \geq 8$, and $c_i(A) \neq c_j(A)$,

(v) for each $i \in N$, $c_i(\mathbb{A}) \geq \max_{A \subset \mathbb{A}} \{c_i(A) + c_i(\mathbb{A} \backslash A)\}$.

Let $A' = (A'_1, A'_2, A'_3) = (\{\alpha\}, \{\beta\}, \{\theta\})$ and $A'' = (\{\beta\}, \{\theta\}, \{\alpha\})$. Note that $W(c) = 7 = \sum_{i \in N} c_i(A'_i) = \sum_{i \in N} c_i(A''_i)$.

By *assignment-efficiency*, $A^\tau(c) \in \{A', A''\}$. Then, $u(\varphi_2^\tau(c); c_2) = -3 + t_2^\tau(c)$ and for $i \in \{1, 3\}$, $u(\varphi_i^\tau(c); c_i) = -2 + t_i^\tau(c)$.

If $A^\tau(c) = A'$, then by *no-envy*, $u(\varphi_1^\tau(c); c_1) \geq -c_1(A_2^\tau(c)) + t_2^\tau(c)$, $u(\varphi_2^\tau(c); c_2) \geq -c_2(A_3^\tau(c)) + t_3^\tau(c)$, and $u(\varphi_3^\tau(c); c_3) \geq -c_3(A_1^\tau(c)) + t_1^\tau(c)$. Since $c_1(A'_1) = c_1(A'_2)$, $c_2(A'_2) = c_2(A'_3)$, and $c_3(A'_3) = c_3(A'_1)$, we have $t_1^\tau(c) \geq t_2^\tau(c) \geq t_3^\tau(c) \geq t_1^\tau(c)$.

If $A^\tau(c) = A''$, then by *no-envy*, $u(\varphi_1^\tau(c); c_1) \geq -c_1(A_3^\tau(c)) + t_3^\tau(c)$, $u(\varphi_2^\tau(c); c_2) \geq -c_2(A_1^\tau(c)) + t_1^\tau(c)$, and $u(\varphi_3^\tau(c); c_3) \geq -c_3(A_2^\tau(c)) + t_2^\tau(c)$. Since $c_1(A''_1) = c_1(A''_3)$, $c_2(A''_2) = c_2(A''_1)$, and $c_3(A''_3) = c_3(A''_2)$, we have $t_1^\tau(c) \geq t_3^\tau(c) \geq t_2^\tau(c) \geq t_1^\tau(c)$.

In both cases, there is $k^\tau \in \mathbb{R}$ such that for each $i \in N$, $t_i^\tau(c) = k^\tau$. By *egalitarian-equivalence* and (5), there are $R(c) \in 2^{\mathbb{A}}$ and $r(c) \in \mathbb{R}$ such that

$$u(\varphi_1^\tau(c); c_1) = -2 + k^\tau = -c_1(R(c)) + r(c). \tag{8}$$

$$u(\varphi_2^\tau(c); c_2) = -3 + k^\tau = -c_2(R(c)) + r(c). \tag{9}$$

$$u(\varphi_3^\tau(c); c_3) = -2 + k^\tau = -c_3(R(c)) + r(c). \tag{10}$$

By (8) and (10), $c_1(R(c)) = c_3(R(c))$. Then, $R(c) \in \{\alpha, \emptyset\}$. Suppose $R(c) = \{\alpha\}$. By (8), $r(c) = k^\tau$. Since $c_2(\{\alpha\}) = 4$, by (9), $-3 + k^\tau = -4 + r(c)$, a contradiction. Suppose $R(c) = \emptyset$. By (8), $r(c) = k^\tau - 2$, which contradicts (9). This completes the proof for part (a). ◇

b) Let $\mathcal{C} \in \{\mathcal{C}_{un}, \mathcal{C}_{ad}, \mathcal{C}_{sub}, \mathcal{C}_{sup}\}$ and consider mechanisms defined on $\bigcup_{N \in \mathcal{N}: |N|=2} \mathcal{C}^N$. Let $\varphi^\tau \equiv (A^\tau, t^\tau)$ be such a mechanism which is *assignment-efficient* and *egalitarian-equivalent* such that for each $N \in \mathcal{N}$ with $|N| = 2$ and each $c \in \mathcal{C}^N$, $R(c) = A_i^\tau(c)$ for some $i \in N$. Assume, by contradiction, that φ^τ is not *envy-free* on $\bigcup_{N \in \mathcal{N}: |N|=2} \mathcal{C}^N$. Then, there is $N = \{i, j\} \in \mathcal{N}$ and $c \in \mathcal{C}^N$ such that $u(A_i^\tau(c), t_i^\tau(c); c_i) < -c_i(A_j^\tau(c)) + t_j^\tau(c)$. That is, by (7),

$$c_i(A_j^\tau(c)) + c_j(R(c)) < c_i(R(c)) + c_j(A_j^\tau(c)). \tag{11}$$

If $R(c) = A_j^\tau(c)$, then by (11), $c_i(A_j^\tau(c)) + c_j(A_j^\tau(c)) < c_i(A_j^\tau(c)) + c_j(A_j^\tau(c))$, a contradiction. If $R(c) = A_i^\tau(c)$, then by (11), $c_i(A_j^\tau(c)) + c_j(A_i^\tau(c)) < c_i(A_i^\tau(c)) + c_j(A_j^\tau(c))$, which contradicts that $(A_i^\tau(c), A_j^\tau(c))$ is an efficient assignment for c. Hence, φ^τ must be *envy-free*. ◇

c) Let $\mathcal{C} \in \{\mathcal{C}_{ad}, \mathcal{C}_{sub}\}$. On $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$, let $\varphi^\tau \equiv (A^\tau, t^\tau)$ be *assignment-efficient* and *welfare-egalitarian*. By Lemma 3d, φ^τ is *egalitarian-equivalent* such that for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $R(c) = \emptyset$. Assume, by contradiction, that φ^τ is not *envy-free* on $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$. Then, there exist $N \in \mathcal{N}$, $c \in \mathcal{C}^N$, and a pair of agents $\{i, j\} \subseteq N$ such that $u(A_i^\tau(c), t_i^\tau(c); c_i) < -c_i(A_j^\tau(c)) + t_j^\tau(c)$. That is, by (7),

$$c_i(A_j^\tau(c)) < c_j(A_j^\tau(c)). \tag{12}$$

This inequality implies that $A_j^\tau(c) \neq \emptyset$. Hence, if $|\mathbb{A}| = 1$, then j is the agent who is assigned the only task which, by (12), contradicts that $A^\tau(c)$ is an efficient assignment for c. If $|\mathbb{A}| \geq 2$, then by (12),

$$c_i(A_i^\tau(c)) + c_i(A_j^\tau(c)) < c_i(A_i^\tau(c)) + c_j(A_j^\tau(c)). \tag{13}$$

Note that on the *subadditive* domain, $c_i(A_i^\tau(c) \cup A_j^\tau(c)) \leq c_i(A_i^\tau(c)) + c_i(A_j^\tau(c))$, which holds as an equality on the *additive* domain. Hence, by (13), it is less costly to assign both $A_i^\tau(c)$ and $A_j^\tau(c)$ to agent i rather than assigning these sets to i and j, respectively. This contradicts that $A^\tau(c)$ is an efficient assignment for c. This completes the proof for part (d). ◇

d) Let $N = \{1, 2, 3\}$, $\mathbb{A} = \{\alpha, \beta, \theta\}$, and $c' \in \mathcal{C}^N_{ad}$ be such that

(i) $c'_1(\{\alpha\}) = c'_1(\{\beta\}) = 10$, $c'_1(\{\theta\}) = 15$,

(ii) $c'_2(\{\alpha\}) = 11$, $c'_2(\{\beta\}) = 12$, $c'_2(\{\theta\}) = 14$,

(iii) $c'_3(\{\alpha\}) = 12$, $c'_3(\{\beta\}) = 11$, $c'_3(\{\theta\}) = 12$.

Let $\mathcal{C} \in \{\mathcal{C}_{ad}, \mathcal{C}_{sub}\}$ and $\varphi^\tau \equiv (A^\tau, t^\tau)$ be an *assignment-efficient* mechanism such that for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$ with $c \neq c'$, and each $i \in N$, $t^\tau_i(c) = \frac{30-W(c)}{n} + c_i(A^\tau_i(c))$; and for each $i \in \{1, 2, 3\}$, $t^\tau_i(c') = c'_i(A^\tau_i(c')) - c'_i(\{\alpha, \theta\}) + 20$. We will show that φ^τ is *envy-free* and *egalitarian-equivalent,* but not *welfare-egalitarian* on $\bigcup\limits_{N \in \mathcal{N}} \mathcal{C}^N$.

It is easy to see that for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$ with $c \neq c'$, φ^τ chooses an *egalitarian-equivalent* allocation with reference set of tasks $R(c) = \emptyset$ and reference transfer $r(c) = \frac{30-W(c)}{n}$. By Proposition 1c, $\varphi^\tau(c)$ is an *envy-free* allocation.

Now, consider c'. By (5), φ^τ chooses an *egalitarian-equivalent* allocation for c' where the reference set is $R(c') = \{\alpha, \theta\}$ and the reference transfer is $r(c') = 20$.

The efficient assignment at c' is $A^\tau(c') = (\{\alpha, \beta\}, \emptyset, \{\theta\})$ and $W(c') = 32$. Agents' transfers at c' are $t^\tau_1(c') = c'_1(A^\tau_1(c')) - c'_1(\{\alpha, \theta\}) + 20 = 15$, $t^\tau_2(c') = -5$, and $t^\tau_3(c') = 8$.

Note that $u(\varphi^\tau_1(c'); c_1) = -c'_1(\{\alpha, \beta\}) + t^\tau_1(c') = -5$; $u(\varphi^\tau_2(c'); c_2) = -c'_2(\emptyset) + t^\tau_2(c') = -5$; and $u(\varphi^\tau_3(c'); c_3) = -c'_3(\{\theta\}) + t^\tau_3(c') = -4$. Since $u(\varphi^\tau_2(c'); c_2) \neq u(\varphi^\tau_3(c'); c_3)$, φ^τ is not *welfare-egalitarian.* Note that

- $u(\varphi^\tau_1(c'); c_1) \geq -c_1(\emptyset) + t^\tau_2(c') = -5$ and
  $u(\varphi^\tau_1(c'); c_1) \geq -c_1(\{\theta\}) + t^\tau_3(c') = -15 + 8$;
- $u(\varphi^\tau_2(c'); c_2) \geq -c_2(\{\alpha, \beta\}) + t^\tau_1(c') = -23 + 15$ and
  $u(\varphi^\tau_2(c'); c_2) \geq -c_2(\{\theta\}) + t^\tau_3(c') = -14 + 8$;
- $u(\varphi^\tau_3(c'); c_3) \geq -c_3(\{\alpha, \beta\}) + t^\tau_1(c') = -23 + 15$ and
  $u(\varphi^\tau_3(c'); c_3) \geq -c_3(\emptyset) + t^\tau_2(c') = -5$.

Hence, $\varphi^\tau(c')$ is an *envy-free* allocation. Altogether, on the *additive* or the *subadditive* domain, φ^τ is *assignment-efficient*, *egalitarian-equivalent*, and *envy-free*, but not *welfare-egalitarian.* ◇ ■

In general, there is no relationship between *welfare-egalitarianism* and *no-envy*. However, by Proposition 1c, we have the following relationship in the multi-object-demand model:

Corollary 1. *On the additive or the subadditive domain or when* $|\mathbb{A}| = 1$, *assignment-efficiency and welfare-egalitarianism together imply no-envy.*

Note that just like in the single-object-demand model, the proofs of Proposition 1 (a) and (b) do not require the empty set to be a reference set of tasks. Note also that the results in Proposition 1b,c,d still hold if we additionally impose *budget-balance*. To see how the proofs in Proposition 1 are modified when we also impose *budget-balance,* consider the following transfers which, by Lemma 3b, balance the budget. Let $T \in \mathbb{R}$. In Proposition 1b, for each $N \in \mathcal{N}$ with $|N| = 2$, each $c \in \mathcal{C}^N$, and each $i \in N$ such that $R(c) = A^\tau_i(c)$, let $t^\tau_i(c) = \frac{1}{2}[T + c_j(A^\tau_i(c)) - c_j(A^\tau_j(c))]$ and $t^\tau_j(c) = \frac{1}{2}[T + c_j(A^\tau_j(c)) - c_j(A^\tau_i(c))]$ where $j = N\backslash\{i\}$. In Proposition 1c, for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$, let $t^\tau_i(c) = c_i(A^\tau_i(c)) + \frac{1}{n}[T - W(c)]$. In Proposition 1d, let $T = 30$ and for each $i \in \{1, 2, 3\}$, let $t^\tau_i(c') = c'_i(A^\tau_i(c')) - c'_i(\{\alpha, \theta\}) + 24$. The proofs would still work with these restrictions on the transfers and the budget would be balanced.

The fact that Proposition 1b,c,d also hold under *budget-balance* demonstrates that we do not need to forgo *budget-balance* or require *strategy-proofness* in order to obtain the compatibility of *no-envy* and *egalitarian-equivalence* in the multi-object-demand model. Hence, *budget-balance* and *strategy-proofness* do not play a role in achieving the compatibility result.

Proposition 1 and Proposition 2 below illustrate the interplay of the three factors which make it possible to obtain compatibility of *no-envy* and *egalitarian-equivalence* in our setting in contrast to the incompatibility result obtained by Thomson (1990) and Chun, Mitra, and Mutuswami (2014), namely,

(i) agents are allowed to be assigned multiple objects,

(ii) costs are *additive* or *subadditive*, and

(iii) the empty set is allowed to be a reference set.

Proposition 1a shows that we do not achieve the compatibility of *no-envy* and *egalitarian-equivalence* by simply allowing agents to be assigned multiple objects. Hence, the multi-object-demand setting requires additional requirements to obtain the compatibility of *no-envy* and *egalitarian-equivalence*. First, comparing Proposition 1a with Proposition 1c,d clarifies that we need the cost functions to be *additive* or *subadditive* in order to obtain the compatibility.

The proof of Proposition 1a still works, if we do not allow the empty set to be a reference set of tasks; hence, the fact that cost functions are either *unrestricted* or *superadditive* is enough to guarantee the incompatibility of *no-envy* and *egalitarian-equivalence*. The proof of Proposition 1a also works, if we require agents to have only single-object-demands and we allow the empty set to be a reference set. Thus, one may think that whether or not we allow the empty set to be a reference set does not play a role in the compatibility of *no-envy* and *egalitarian-equivalence*. This line of thought would not be correct as Proposition 2 below demonstrates: on the *additive* or the *subadditive* domain, if we do not allow the empty set to be a reference set, then there is no *assignment-efficient*, *egalitarian-equivalent*, and *envy-free* mechanism.

In conclusion, in order to obtain compatibility of *no-envy* and *egalitarian-equivalence* in the problem of allocating heterogeneous objects and money, we need all of three factors (i), (ii) and (iii) at the same time, one without the other is not enough to guarantee the compatibility.

Proposition 2. *On the additive or the subadditive domain, if an assignment-efficient and egalitarian-equivalent mechanism is envy-free, then there are $N \in \mathcal{N}$ and $c \in \mathcal{C}^N$ with $\mathcal{C} \in \{\mathcal{C}_{ad}, \mathcal{C}_{sub}\}$ such that $R(c) = \emptyset$.*

Proof: Let $\mathcal{C} \in \{\mathcal{C}_{ad}, \mathcal{C}_{sub}\}$. On $\bigcup_{N\in\mathcal{N}} \mathcal{C}^N$, let $\varphi^\tau \equiv (A^\tau, t^\tau)$ be *assignment-efficient* and *egalitarian-equivalent*. Assume that for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $R(c) \neq \emptyset$. We will show that φ^τ is not *envy-free* on $\bigcup_{N\in\mathcal{N}} \mathcal{C}^N$.

Let $N = \{1, 2, 3\}$, $\mathbb{A} = \{\alpha, \beta, \theta\}$, and $c' \in \mathcal{C}^N_{ad}$ be as in Proposition 1d. Let $\widehat{c}_3 \in \mathcal{C}_{ad}$ be such that $\widehat{c}_3(\{\alpha\}) = 14$, $\widehat{c}_3(\{\beta\}) = 15$, $\widehat{c}_3(\{\theta\}) = 12$. Let $\widehat{c} \in \mathcal{C}^N_{ad}$ be such that $\widehat{c} = (\widehat{c}_3, c'_{-3})$. Note that $A^\tau_1(\widehat{c}) = \{\alpha, \beta\}$, $A^\tau_2(\widehat{c}) = \emptyset$, $A^\tau_3(\widehat{c}) = \{\theta\}$, and $W(\widehat{c}) = 32$. Assume, by contradiction, that φ^τ is *envy-free* on $\bigcup_{N\in\mathcal{N}} \mathcal{C}^N$. Then, by (7), for $j = 2$ and for each $i \in \{1, 3\}$, $\widehat{c}_i(A^\tau_j(\widehat{c})) + \widehat{c}_j(R(\widehat{c})) \geq \widehat{c}_j(A^\tau_j(\widehat{c})) + \widehat{c}_i(R(\widehat{c}))$. This inequality and $A^\tau_2(\widehat{c}) = \emptyset$ together imply that for each $i \in \{1, 3\}$,

$$\widehat{c}_2(R(\widehat{c})) \geq \widehat{c}_i(R(\widehat{c})). \tag{14}$$

Note that $\widehat{c}_2(\{\theta\}) < \widehat{c}_1(\{\theta\})$ and for each $A \in (2^{\mathbb{A}} \setminus \{\emptyset, \{\theta\}\})$, $\widehat{c}_2(A) < \widehat{c}_3(A)$. These inequalities and the fact that $R(\widehat{c}) \neq \emptyset$ together contradict (14). ■

Note that the proof of Proposition 2 still works if we additionally require φ^τ to be *budget-balanced*. That is, by Lemma 3d, on the *additive* or the *subadditive* domain, if a mechanism is *Pareto-efficient*, *egalitarian-equivalent,* and *envy-free,* then it has to pick a *welfare-egalitarian* allocation for some economies. On the other hand, by Proposition 1d, on the *additive* or the *subadditive* domain, an *assignment-efficient* (or *Pareto-efficient*), *egalitarian-equivalent*, and *envy-free* mechanism does not need to be a *welfare-egalitarian* mechanism.

3.2 Characterizations Under Strategy-Proofness

The next question we investigate is how the results in Propositions 1 and 2 would change when we also impose *strategy-proofness*. Fortunately, in the multi-object-demand model, when costs are *subadditive* or *additive*, *assignment-efficiency*, *no-envy,* and *egalitarian-equivalence* are still compatible under *strategy-proofness* since not only the *Egalitarian mechanisms,* but also the following Groves mechanisms satisfy *no-envy* and *egalitarian-equivalence*.

The *Extended-Egalitarian* mechanism associated with $\gamma \in \Gamma$ and $\tau \in \mathcal{T}$, $\widehat{E}^{\gamma,\tau}$:
Let $\widehat{E}^{\gamma,\tau} \equiv (A^\tau, \widehat{t}^{\,\gamma,\tau})$ be such that for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $(A_i^\tau(c))_{i\in N}$ is an efficient-assignment for c; and for each $N \in \mathcal{N}$ and each $i \in N$,

if $|N| > 2$,

$$\widehat{t}_i^{\,\gamma,\tau}(c) = t_i^{\gamma,\tau}(c) \text{ for each } c \in \mathcal{C}^N, \tag{15}$$

if $|N| = 2$,

$$\text{(i) either } \widehat{t}_i^{\,\gamma,\tau}(c) = t_i^{\gamma,\tau}(c) \text{ for each } c \in \mathcal{C}^N, \tag{16}$$

$$\text{(ii) or } \widehat{t}_i^{\,\gamma,\tau}(c) = c_j(\mathbb{A}) - c_j(A_j^\tau(c)) + \gamma(N), \text{ where } j \in N\backslash\{i\}, \text{ for each } c \in \mathcal{C}^N. \tag{17}$$

Let $\widehat{\mathcal{E}} \equiv \{\widehat{E}^{\gamma,\tau} |\ \gamma \in \Gamma, \tau \in \mathcal{T}\}$ be the class of such mechanisms. Note that $\mathcal{E} \subset \widehat{\mathcal{E}}$.

Note also that $\widehat{E}^{\gamma,\tau} = G^{h,\tau}$ such that for each $N \in \mathcal{N}$ and each $i \in N$,

$$\text{if } |N| > 2, \text{ then } h_i(c_{-i}) = \gamma(N) \text{ for each } c \in \mathcal{C}^N, \tag{18}$$

$$\text{if } |N| = 2, \text{ then}$$

$$\text{(i) either } h_i(c_{-i}) = \gamma(N) \text{ for each } c \in \mathcal{C}^N, \tag{19}$$

$$\text{(ii) or } h_i(c_{-i}) = \gamma(N) + c_j(\mathbb{A}), \text{ where } j \in N\backslash\{i\}, \text{ for each } c \in \mathcal{C}^N. \tag{20}$$

That is, when there are more than two agents, the transfers of $\widehat{E}^{\gamma,\tau}$ are same as the transfers of an *Egalitarian mechanism* $E^{\gamma,\tau}$. However, when there are only two agents, then the mechanism has two options for transfers, option (i) or (ii). For some pairs of agents, the transfers can be chosen equal to the transfers of $E^{\gamma,\tau}$ (option (i)); and for other pairs, the transfers can be chosen to be as in (ii). Note that for a given two-agent population N, for all economies c that pertain to population N, the mechanism should stick to one type of transfer: either (i) or (ii). That is, for a given $N \in \mathcal{N}$ with $|N| = 2$, either (i) applies for each $c \in \mathcal{C}^N$ or, (ii) applies for each $c \in \mathcal{C}^N$.

When there are only two agents, the transfers specified by $\widehat{E}^{\gamma,\tau}$ in option (ii) differ from the transfers of a Pivotal (Vickrey) mechanism by the amount $\gamma(N)$. Consider $\gamma \in \Gamma$ such that for each $N \in \mathcal{N}$ with $|N| = 2$, $\gamma(N) = 0$. Let the mechanism $\widehat{E}^{\gamma,\tau}$ choose option (ii) for the transfers in all two-agent economies. Then, $\widehat{E}^{\gamma,\tau}$ is such that for each economy with at least three agents, the transfers of $\widehat{E}^{\gamma,\tau}$ are same as the transfers of an *Egalitarian mechanism* $E^{\gamma,\tau}$; and for each two-agent economy, the transfers of $\widehat{E}^{\gamma,\tau}$ are that of a Pivotal mechanism. Such a mechanism belongs to $(\widehat{\mathcal{E}}\backslash\mathcal{E})$.

By Proposition 1, *assignment-efficient, egalitarian-equivalent,* and *envy-free* mechanisms exist only on the *additive* or the *subadditive* domain. Also, as in Proposition 1d, we can construct infinitely many such mechanisms, each satisfying (5) and (7), and choosing possibly different sets of tasks as reference sets in different economies. The aforementioned axioms and the features of the multi-object-demand model provide enough degree of freedom to do so. If we additionally impose *strategy-proofness*, would we introduce enough restriction to the model to eliminate the multiplicity of possible mechanisms satisfying *assignment-efficiency*, *egalitarian-equivalence*, and *no-envy*? Our Theorem next answers this question in an affirmative way. Without *strategy-proofness*, we have some freedom to choose different sets of tasks as reference sets in different economies. *Strategy-proofness* links different economies with the same agent set and cuts back this freedom: for each population $N \in \mathcal{N}$, only certain sets of tasks can act as a reference set. This is due to the fact that by (3), for each $N \in \mathcal{N}$ and each agent $i \in N$, $i's$ transfer in a given economy $c \in \mathcal{C}^N$ also depends on her transfers in other economies $c' \in \mathcal{C}^N$ where $c'_{-i} = c_{-i}$; and in each c, agents' transfers restrict the choice of $R(c)$ through (5) and (7).

Theorem 1. *On the subadditive domain, an assignment-efficient and strategy-proof mechanism is egalitarian-equivalent and envy-free if and only if it belongs to $\widehat{\mathcal{E}}$.*

The proofs of all the theorems are in the Appendix.

All our proofs in this Subsection also work if we restrict our selves to the *additive* domain or to the single-task case. Note that on the *superadditive* or the *unrestricted* domain, by Yengin (2012b), the mechanisms in $\widehat{\mathcal{E}}$ are still *egalitarian-equivalent*, which by Proposition 1a, implies that they are not *envy-free* on these domains. In fact, by Pápai (2003), on the *unrestricted* domain, no Groves mechanism is *envy-free*.

Consider the *subadditive* domain. The proof of Theorem 1 shows that, under *assignment efficiency* and *strategy-proofness*, *egalitarian-equivalence* and *no-envy* together indicate that only the empty set can be chosen as the reference set in any economy with at least three agents; and for each population $N \in \mathcal{N}$ with $|N| = 2$, either the empty set should be chosen as the reference set in each economy $c \in \mathcal{C}^N$; or the set of all tasks, $\mathbb{A}$, should be chosen as the reference set in each $c \in \mathcal{C}^N$. The next result follows from Lemma 3d and Theorem 1.

Corollary 2. *On the subadditive domain of economies containing at least three agents,* $\bigcup_{N \in \mathcal{N}:|N|\geq 3} \mathcal{C}^N_{sub}$, *under assignment efficiency and strategy-proofness, egalitarian-equivalence together with no-envy is equivalent to welfare-egalitarianism.*

Suppose new agents join some initial population. The cost of an efficient assignment in the larger population is at most as large as the one in the smaller population. Hence, a population increase is good news for the society. Since none of the agents in the initial population is responsible for the population increase, all of them should be at least as well off in the larger population as in the smaller one (Thomson, 1983).

Population Monotonicity: For each pair $\{N, N'\} \subset \mathcal{N}$ such that $N' \subset N$, each $i \in N'$, and each $c \in \mathcal{C}^N$,

$$u(\varphi_i(c); c_i) \geq u(\varphi_i(c_{N'}); c_i).$$

Lemma 4. *A Groves mechanism $G^{h,\tau}$ is population monotonic if and only if for each pair $\{N, N'\} \subset \mathcal{N}$ such that $N' \subset N$, each $i \in N'$, and each $c \in \mathcal{C}^N$,*

$$h_i(c_{-i}) \geq h_i(c_{N' \setminus \{i\}}). \tag{21}$$

Proof: Let $h \in \mathcal{H}$ be as in (21). Let $\{N, N'\} \subset \mathcal{N}$ be such that $N' \subset N$, $i \in N'$, and $c \in \mathcal{C}^N$. Since $W(c_{N'}) \geq W(c)$, by (4) and (21), $u(G_i^{h,\tau}(c); c_i) \geq u(G_i^{h,\tau}(c_{N'}); c_i)$. Hence, $G^{h,\tau}$ is *population monotonic*.

Conversely, let $G^{h,\tau}$ be a *population monotonic* Groves mechanism. Assume, by contradiction, that there are $\{N, N'\} \subset \mathcal{N}$ such that $N' \subset N$, $i \in N'$, and $c \in \mathcal{C}^N$ for which

$$h_i(c_{-i}) < h_i(c_{N' \setminus \{i\}}). \tag{22}$$

Let $\widehat{c}_i \in \mathcal{C}$ be such that for each $A \in (2^{\mathbb{A}} \setminus \emptyset)$, $\widehat{c}_i(A) = 0$. Consider $\widehat{c} = (\widehat{c}_i, c_{-i}) \in \mathcal{C}^N$. Note that $W(\widehat{c}) = W(\widehat{c}_{N'}) = 0$. Hence, by (4), $u(G_i^{h,\tau}(\widehat{c}); \widehat{c}_i) = h_i(\widehat{c}_{-i})$ and $u(G_i^{h,\tau}(\widehat{c}_{N'}); \widehat{c}_i) = h_i(\widehat{c}_{N' \setminus \{i\}})$. Since $\widehat{c}_{-i} = c_{-i}$ and $\widehat{c}_{N' \setminus \{i\}} = c_{N' \setminus \{i\}}$, by (22), $u(G_i^{h,\tau}(\widehat{c}); \widehat{c}_i) < u(G_i^{h,\tau}(\widehat{c}_{N'}); \widehat{c}_i)$, which contradicts *population monotonicity*. □

Our next Theorem states that if we add *population monotonicity* in Theorem 1, then the class $(\widehat{\mathcal{E}} \setminus \mathcal{E})$ is ruled out completely and we are left with a subclass of $\mathcal{E}$.

Theorem 2. *On the subadditive domain, an assignment-efficient and strategy-proof mechanism is egalitarian-equivalent, envy-free, and population monotonic if and only if it is an Egalitarian mechanism in $\mathcal{E}^\gamma$ where $\gamma : \mathcal{N} \to \mathbb{R}$ is such that for each pair $\{N', N\} \subset \mathcal{N}$ with $N' \subset N$,*

$$\gamma(N) \geq \gamma(N'). \tag{23}$$

Lemma 1 and Theorem 2 together imply the following relation.

Corollary 3. *On the subadditive domain of economies containing any number of agents, under assignment efficiency and strategy-proofness, egalitarian-equivalence, no-envy, and population monotonicity together imply welfare-egalitarianism.*

Theorem 2 also brings good news in showing the compatibility of *population monotonicity* and *no-envy*, which exists neither in the single-object-demand model of allocating heterogeneous objects and money (Alkan, 1994; Tadenuma and Thomson, 1993) nor in most other models: for instance, in the problem of allocating an infinitely divisible good over which agents have single-peaked preferences (Thomson, 1995) or in exchange economies when *Pareto-efficiency* is required (Kim, 2004).

3.3 The Pareto-undominated Egalitarian Mechanisms

Let $N \in \mathcal{N}$ and $c \in \mathcal{C}^N$. Allocation $(A_i, t_i)_{i \in N}$ *Pareto-dominates* $(A'_i, t'_i)_{i \in N}$ for c if and only if for each $i \in N$, $u(A_i, t_i; c_i) \geq u(A'_i, t'_i; c_i)$ with strict inequality for some $i \in N$. The mechanism φ *Pareto-dominates* φ' if for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$, $u(\varphi_i(c); c_i) \geq u(\varphi'_i(c); c_i)$ and there are $N \in \mathcal{N}$, $c \in \mathcal{C}^N$, and $i \in N$ such that $u(\varphi_i(c); c_i) > u(\varphi'_i(c); c_i)$.

Since in our model, there is no upper bound on each agent's transfer or the total transfer, every mechanism is Pareto-dominated by another one which gives higher transfers to the agents. However, the latter mechanism would generate a larger deficit for the center compared to the former one. In other words, in our model, there is no Pareto-dominant or Pareto-undominated mechanism unless we impose an upper bound on the total transfer. To be able to make meaningful welfare comparisons, we consider Groves mechanisms that respect a given upper bound $T \in \mathbb{R}$ on total transfer as in Ohseto (2004, 2006). Even though, we can not have *Pareto-efficient* Groves mechanisms, fortunately, Groves mechanisms including the *Egalitarian* mechanisms that satisfy *$T-$bounded-deficit* exist.

Lemma 5. *For each $T \in \mathbb{R}$, an Egalitarian mechanism $E^{\gamma,\tau}$ satisfies $T-$bounded-deficit if and only if for each $N \in \mathcal{N}$, $\gamma(N) \leq \frac{T}{n}$.*

Proof: Let $T \in \mathbb{R}$. $E^{\gamma,\tau}$ satisfies $T-$*bounded-deficit* if and only if for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum_{i\in N} t_i^{\gamma,\tau}(c) = -(n-1)W(c) + n\gamma(N) \leq T$. That is, for each $N \in \mathcal{N}$, $\gamma(N) \leq \min_{c\in\mathcal{C}^N}\{\frac{T+(n-1)W(c)}{n}\} = T/n$. Note that the right-hand-side-of the inequality achieves its minimal value at an economy $c \in \mathcal{C}^N$ where $W(c) = 0$ (for instance, if $c_i(\mathbb{A}) = 0$ for some $i \in N$). □

Definition 1. For each $T \in \mathbb{R}$, let
(i) $\mathcal{E}^{T-BD} \equiv \{E^{\gamma,\tau} | \ \gamma : \mathcal{N} \to \mathbb{R}$ is such that for each $N \in \mathcal{N}$, $\gamma(N) \leq \frac{T}{n}$, $\tau \in \mathcal{T}\}$ be the class of *Egalitarian* mechanisms that satisfy $T-$*bounded-deficit*.
(ii) $\gamma^T : \mathcal{N} \to \mathbb{R}$ be such that for each $N \in \mathcal{N}$, $\gamma^T(N) = \frac{T}{n}$ and $\mathcal{E}^{\gamma^T} \equiv \{E^{\gamma^T,\tau} | \ \tau \in \mathcal{T}\} \subset \mathcal{E}^{T-BD}$.
(iii) $\widehat{\mathcal{E}}^{T-BD} \equiv \{\widehat{E}^{\gamma,\tau} |$ for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum_{i\in N} \widehat{t}_i^{\gamma,\tau}(c) \leq T$, $\tau \in \mathcal{T}\}$ be the class of *Extended-Egalitarian* mechanisms that satisfy $T-$*bounded-deficit*.

We will show that for each $T \in \mathbb{R}$, the *Egalitarian* mechanisms in $\mathcal{E}^{\gamma^T}$ are the "*best*" *Egalitarian* mechanisms in $\mathcal{E}^{T-BD}$ in the sense that, under $T-$*bounded-deficit*, they constitute the Pareto-dominant class among several subclasses of Groves mechanisms satisfying fairness axioms and they are Pareto-undominated by any other Groves mechanism (Theorem 3).

The next result shows that an *Extended-Egalitarian* mechanism satisfies $T-$*bounded-deficit* if and only if it is an *Egalitarian* Groves mechanism satisfying $T-$*bounded-deficit*. In other words, if we impose $T-$*bounded-deficit* on $\widehat{\mathcal{E}}$, then the mechanisms in $\widehat{\mathcal{E}}\backslash\mathcal{E}$ are ruled out.

Proposition 3. *Let $T \in \mathbb{R}$.*
a) $\widehat{\mathcal{E}}^{T-BD} = \mathcal{E}^{T-BD}$.

b) *On the subadditive domain, $\mathcal{E}^{T-BD}$ is the class of all egalitarian-equivalent and envy-free Groves mechanisms that satisfy $T-$bounded-deficit.*

Proof: Let $T \in \mathbb{R}$.
a) Let $\widehat{E}^{\gamma,\tau} \in \widehat{\mathcal{E}}^{T-BD}$. Assume, by contradiction, that $\widehat{E}^{\gamma,\tau} \in \widehat{\mathcal{E}}^{T-BD}\backslash\mathcal{E}^{T-BD}$. Then, by (17), there is $N = \{i,j\} \in \mathcal{N}$ such that $\widehat{t}_i^{\gamma,\tau}(c) = c_j(\mathbb{A}) - c_j(A_j^\tau(c)) + \gamma(N)$ and $\widehat{t}_j^{\gamma,\tau}(c) = c_i(\mathbb{A}) - c_i(A_i^\tau(c)) + \gamma(N)$ for each $(c_i, c_j) \in \mathcal{C}^N$. Let $c' = (c'_i, c'_j) \in \mathcal{C}^N$ be such that $c'_i(\mathbb{A}) > T - 2\gamma(N)$ and $c'_j(\mathbb{A}) = 0$. Since $W(c') = c'_j(\mathbb{A}) = 0$, $\sum_{l\in N} \widehat{t}_l^{\gamma,\tau}(c') = c'_i(\mathbb{A}) + 2\gamma(N) > T$, which contradicts $T-$*bounded-deficit*.

b) On $\bigcup_{N\in\mathcal{N}} \mathcal{C}^N_{sub}$, let $G^{h,\tau}$ be an *egalitarian-equivalent* and *envy-free* Groves mechanism that satisfies $T-$*bounded-deficit*. By Theorem 1 and Definition 1iii, $G^{h,\tau} \in \widehat{\mathcal{E}}^{T-BD}$. By part (a), $G^{h,\tau} \in \mathcal{E}^{T-BD}$.■

The next Corollary follows from Lemma 1 and Proposition 3b.

Corollary 4. *Let $T \in \mathbb{R}$. On the subadditive domain, under assignment efficiency, strategy-proofness, and $T-$bounded-deficit, egalitarian-equivalence together with no-envy is equivalent to welfare-egalitarianism.*

It is easy to see that, by (1), the *Egalitarian* mechanisms in $\mathcal{E}^{\gamma^T}$ Pareto-dominate any other *Egalitarian* mechanism that satisfies $T-$*bounded-deficit*. Moreover, by Proposition 3a, $\mathcal{E}^{\gamma^T}$ is also the Pareto-dominant class in $\widehat{\mathcal{E}}^{T-BD}$. In other words, by Proposition 3b, on the *subadditive* domain, $\mathcal{E}^{\gamma^T}$ is the Pareto-dominant class within the class of all *egalitarian-equivalent* and *envy-free* Groves mechanisms that satisfy $T-$*bounded-deficit* (Theorem 3a). What if we let go of *egalitarian-equivalence* and/or *no-envy?* In that case, could we obtain any Groves mechanism satisfying $T-$*bounded-deficit* which Pareto-dominates the *Egalitarian* mechanisms in $\mathcal{E}^{\gamma^T}$? We show that the answer is negative as explained below:

First, suppose that we drop *no-envy*. By Proposition 2 in Yengin (2012b), an *egalitarian-equivalent* Groves mechanism $G^{h,\tau}$ satisfies $T-$*bounded-deficit* if and only if $G^{h,\tau} \in \mathcal{E}^{T-BD}$. That is, on the *subadditive* domain, *even though, the class of* egalitarian-equivalent *Groves mechanisms is strictly larger than the class of* envy-free *and* egalitarian-equivalent *Groves mechanisms* $\widehat{\mathcal{E}}$*, when we impose* $T-$*bounded-deficit, we end up with the same subclass* $\mathcal{E}^{T-BD}$. Thus, $\mathcal{E}^{\gamma^T}$ is the Pareto-dominant class among all *egalitarian-equivalent* Groves mechanisms that satisfy $T-$*bounded-deficit* (Theorem 3b).

Secondly, suppose that we let go of *egalitarian-equivalence* and keep *no-envy.* In the same model as ours, Pápai (2003) characterizes, on the *subadditive* domain, the class of *envy-free* Groves mechanisms. However, she does not study the implications of imposing $T-$*bounded-deficit.* On the other hand, Yengin (2012a) shows that on the *additive* or the *subadditive* domain, $\mathcal{E}$ is the class of all *envy-free* Groves mechanisms that satisfy *solidarity.* Hence, on these domains, $\mathcal{E}^{\gamma^T}$ is the Pareto-dominant class among all *envy-free* Groves mechanisms that satisfy *solidarity* and $T-$*bounded-deficit* (Theorem 3c).

Ohseto (2006) studies a single-object-demand model where there are $|\mathbb{A}| = s$ $(1 \leq s \leq n)$ number of homogenous objects. Although, his model is different than ours, for the case of $|\mathbb{A}| = 1$, we can apply the results in Ohseto (2006) to our setting and show that, by Theorem 2 in Ohseto (2006), $\mathcal{E}^{\gamma^T}$ belongs to the Pareto-undominated class within the class of all *envy-free* Groves mechanisms that satisfy $T-$*bounded-deficit.*

Let $|\mathbb{A}| = 1$ and $\mathcal{F}^1$ be the class of all *envy-free* Groves mechanisms that satisfy $T-$*bounded-deficit.* Let $\mathcal{F}^2 \subset \mathcal{F}^1$ be the set of Pareto-undominated mechanisms in $\mathcal{F}^1$. That is, for any $G^{h,\tau} \in \mathcal{F}^2$, there is no mechanism in $\mathcal{F}^1$ that Pareto-dominates $G^{h,\tau}$. By Definition 2 in Ohseto (2006) applied to our setting, let $p \geq 0$ and $G^{p,\tau}$ be a Groves mechanism such that for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$,

$$\begin{aligned} \text{if } A_i^\tau(c) &= \mathbb{A}, \text{ then } t_i^{p,\tau}(c) = \frac{T-(n-1)p}{n} + \min\{0, W(c_{-i}) + p\}; \\ \text{if } A_i^\tau(c) &= \emptyset, \text{ then } t_i^{p,\tau}(c) = \frac{T+p}{n} - \max\{0, W(c_{-i}) + p\}. \end{aligned}$$

By Theorem 2 in Ohseto (2006), $\mathcal{F}^2 = \{G^{p,\tau} : p \geq 0,\ \tau \in \mathcal{T}\}$. Note that if $A_i^\tau(c) = \mathbb{A}$, then $-\sum\limits_{j \in N\setminus\{i\}} c_j(A_j^\tau(c)) = 0$; and if $A_i^\tau(c) = \emptyset$, then $-\sum\limits_{j \in N\setminus\{i\}} c_j(A_j^\tau(c)) = -W(c_{-i})$. Hence, $G^{p,\tau}$ for $p = 0$ is such that for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$, $t_i^{p,\tau}(c) = -\sum\limits_{j \in N\setminus\{i\}} c_j(A_j^\tau(c)) + \frac{T}{n}$. By (2), $G^{0,\tau} = E^{\gamma^T,\tau}$. Hence, $\mathcal{E}^{\gamma^T} \subset \mathcal{F}^2$. In other words, when $|\mathbb{A}| = 1$, there is no *envy-free* Groves mechanism that satisfies $T-$*bounded-deficit* and Pareto-dominates the *Egalitarian* mechanisms in $\mathcal{E}^{\gamma^T}$.

Now, consider the multi-object-demand model where $|\mathbb{A}| > 1$. Is $\mathcal{E}^{\gamma^T}$ still the Pareto-undominated class among all *envy-free* Groves mechanisms that satisfy $T-$*bounded-deficit*? Actually, the class $\mathcal{E}^{\gamma^T}$ achieves more than that. Theorem 3f shows that on any domain, $\mathcal{E}^{\gamma^T}$ is the Pareto-undominated class within the class of all Groves mechanisms that satisfy $T-$*bounded-deficit*.[10]

Finally, we consider certain subclasses of the class of Groves mechanisms that satisfy alternative fairness notions and $T-$*bounded-deficit*. It turns out that $\mathcal{E}^{\gamma^T}$ is also the Pareto-dominant class within these subclasses (Theorem 3d,e).

Theorem 3. *Let $T \in \mathbb{R}$.*

a) *The mechanisms in $\mathcal{E}^{\gamma^T}$ Pareto-dominate every mechanism in $\widehat{\mathcal{E}}^{T-BD} \setminus \mathcal{E}^{\gamma^T}$.*

b) *Among all assignment-efficient, strategy-proof, and egalitarian-equivalent mechanisms that satisfy $T-$bounded-deficit, the mechanisms in $\mathcal{E}^{\gamma^T}$ Pareto-dominate the others and generate the minimal surplus.*

c) *On the subadditive domain, among all assignment-efficient, strategy-proof, and envy-free mechanisms that satisfy solidarity and $T-$bounded-deficit, the mechanisms in $\mathcal{E}^{\gamma^T}$ Pareto-dominate the others and generate the minimal surplus.*

d) *Among all assignment-efficient, strategy-proof, and order preserving mechanisms that satisfy solidarity and $T-$bounded-deficit, the mechanisms in $\mathcal{E}^{\gamma^T}$ Pareto-dominate the others and generate the minimal surplus.*

e) *Among all assignment-efficient, strategy-proof, and unanimous mechanisms that satisfy solidarity and $T-$bounded-deficit, the mechanisms in $\mathcal{E}^{\gamma^T}$ Pareto-dominate the others.*

f) *The mechanisms in $\mathcal{E}^{\gamma^T}$ are Pareto-undominated by any other Groves mechanism that satisfies $T-$bounded-deficit.*

Theorem 3c indicates the compatibility of *solidarity* and *no-envy* in our model. Contrast this result with the impossibility result in queueing problems where no *assignment-efficient, budget-balanced,* and *envy-free* mechanism satisfies *solidarity* (Chun, 2006).

The following set of deficit upper bounds are linear functions of the total cost of an efficient assignment:

$$\mathcal{M} = \{m^{k,T} : \bigcup_{N\in\mathcal{N}} \mathcal{C}^N \to \mathbb{R} |\ \forall\, c \in \bigcup_{N\in\mathcal{N}} \mathcal{C}^N,\ m^{k,T}(c) = kW(c) + T \text{ with } k \geq -(n-1) \text{ and } T \in \mathbb{R}\}.$$

For each $m^{k,T} \in \mathcal{M}$, a mechanism satisfies $m^{k,T}-$*bounded-deficit* if for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum\limits_{i\in N} t_i(c) \leq m^{k,T}(c)$. One intuitive example of $\mathcal{M}$ is the upper bound that requires total compensation to agents not to exceed the total cost they incur ($k = 1$, $T = 0$) : for each $c \in \bigcup\limits_{N\in\mathcal{N}} \mathcal{C}^N$, $\sum\limits_{i\in N} t_i(c) \leq W(c)$. Note that $m^{k,T}-$*bounded-deficit* with $k = 0$ is $T-$*bounded-deficit;* and with $k = T = 0$ is *no-deficit*.

It is easy to see that Lemma 5 still holds if $T-$*bounded-deficit* is replaced with $m^{k,T}-$*bounded-deficit*.[11] That is, $\mathcal{E}^{T-BD}$ is also the class of *Egalitarian* mechanisms that satisfy $m^{k,T}-$*bounded-deficit*. Hence, all the proofs of Theorem 3 would also work if we replace $T-$*bounded-deficit* with $m^{k,T}-$*bounded-deficit*.

[10] Hence, under $T-$*bounded-deficit*, the Pareto-dominant class among all *egalitarian-equivalent* Groves mechanisms are Pareto-undominated by any (*envy-free*) Groves mechanism. Contrast this with Ohseto (2004, 2006) who shows that in the single-object-demand model, there are members of $\mathcal{F}^2$ which Pareto-dominate those Groves mechanisms that belong to the Pareto-dominant class within the class of all *egalitarian-equivalent* Groves mechanisms satisfying $T-$*bounded-deficit*. This is no longer the case in our multi-object-demand model.

[11] $E^{\gamma,\tau}$ satisfyies $m^{k,T}-$*bounded-deficit* if and only if for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $\sum\limits_{i\in N} t_i^{\gamma,\tau}(c) = -(n-1)W(c) + n\gamma(N) \leq kW(c) + T$. That is, for each $N \in \mathcal{N}$, $\gamma(N) \leq \min\limits_{c\in\mathcal{C}^N} \{\frac{T+(k+n-1)W(c)}{n}\} = T/n$.

Obviously, if we give up *strategy-proofness*, then there are mechanisms within the class of all *assignment-efficient*, *egalitarian-equivalent,* and *envy-free* mechanisms that satisfy $T-$*bounded-deficit* and Pareto-dominate the mechanisms in $\mathcal{E}^{\gamma^T}$ by providing agents higher transfers. For instance, under $T-$*bounded-deficit* where $T = 30$, the mechanism in Proposition 1d Pareto-dominates the mechanisms in $\mathcal{E}^{\gamma^T}$ while generating higher deficits in each economy. When a *Pareto-efficient* (where total transfer is T), *egalitarian-equivalent,* and *envy-free* mechanism is chosen over the ones in $\mathcal{E}^{\gamma^T}$, the increase in total welfare of agents is exactly equal to the increase in total deficit. Also, if we give up *strategy-proofness*, then due to the manipulation of the mechanism by the agents, it may be impossible to achieve *assignment-efficient*, *egalitarian-equivalent,* and *envy-free* allocations in the first place. Thus, under $T-$*bounded-deficit*, assuming that *strategy-proofness* is essential to achieve *assignment-efficiency*, *egalitarian-equivalency,* and *no-envy,* for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, the minimum cost of this achievement for the center is the deficit $\sum\limits_{i \in N} t_i^{\gamma^T,\tau}(c) = -(n-1)W(c) + T$.

Finally, lets investigate whether there are *strategy-proof* mechanisms that satisfy $T-$*bounded-deficit* and Pareto-dominate the mechanisms in $\mathcal{E}^{\gamma^T}$. In the multi-object-demand model, characterization of the class of *strategy-proof* and fair mechanisms is a complicated question which hasn't been addressed so far. The following class of mechanisms (adapted to our variable population and task allocation setting) are *strategy-proof* (Proposition 1.31 in Nisan et. al., 2007).

For each $N \in \mathcal{N}$, let $\omega^N = (\omega_i)_{i \in N}$ be the vector of weights for population N such that for each $i \in N$, $\omega_i \geq 0$ with strict inequality for some $i \in N$. Let $\omega = (\omega^N)_{N \in \mathcal{N}}$ and $\mathcal{W}$ be the set of all ω. Let $F : \bigcup\limits_{N \in \mathcal{N}} \mathcal{A}(N) \to \mathbb{R}$ be a function that associates each assignment for each population with a real number. Let $\mathcal{F}$ be the set of all such functions. Let $s = (\omega, F, h, \tau) \in \mathcal{S} = \mathcal{W} \times \mathcal{F} \times \mathcal{H} \times \mathcal{T}$.

The *Weighted-Groves* mechanism associated with $s = (\omega, F, h, \tau) \in \mathcal{S}$, G^s:
Let $G^s \equiv (A^s, t^s)$ be a mechanism such that for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$,

$$A^s(c) \in \arg\min \left\{ \sum_{j \in N} \omega_j^N c_j(A'_j) + F(A') : A' = (A'_j)_{j \in N} \in \mathcal{A}(N) \right\} \tag{24}$$

and for each $i \in N$ with $\omega_i^N > 0$,

$$t_i^s(c) = -\frac{1}{\omega_i^N} \left[\sum_{j \in N \setminus \{i\}} \omega_j^N c_j(A^s(c)) + F(A^s(c)) \right] + h_i(c_{-i}). \tag{25}$$

Let $\mathcal{G}$ be the set of *Weighted-Groves* mechanisms. Note that for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$ with $\omega_i^N = 0$, c_i is ignored when calculating $A^s(c)$ and hence, any arbitrary t_i can be chosen without effecting the *strategy-proofness* of G^s.[12] By Mishra and Sen (2012), if a *Weighted-Groves* mechanism G^s is *neutral* (mechanism should treat all alternatives symmetrically), then for each $N \in \mathcal{N}$ and each $\{A', A''\} \subseteq \mathcal{A}(N)$, $F(A') = F(A'')$ in which case F can be suppressed in (24) and (25) by simply assuming that for each $N \in \mathcal{N}$ and each $A' \in \mathcal{A}(N)$, $F(A') = 0$. Let $\mathcal{G}^N$ be the class of *neutral Weighted-Groves* mechanisms. Note that if G^s is *neutral* and gives equal weights to all potential agents, then G^s is a Groves mechanism.

In his seminal paper, Roberts (1979) showed that $\mathcal{G}$ is the class of all *strategy-proof* mechanisms when there is no restriction on the domain of agents' valuations/costs and the set of outcomes is finite and has at least three elements. Note that the domain of preferences in the problem of

[12] If for each $N \in \mathcal{N}$, there is $i \in N$ such that $\omega_i^N = 0$, then i can be made to absorb any surplus or deficit generated at any $c \in \mathcal{C}^N$ (i.e., budget would be balanced). If for each $N \in \mathcal{N}$ and each $i \in N$, $\omega_i^N > 0$, then G^s is not *budget-balanced.*

allocating objects and money belongs to the auction domains, which are restricted domains since agents have no externalities and their valuation of an assignment depends only on what set of objects they receive in that assignment. Mishra and Sen (2012) state that there has been no "functional form" characterization of implementable social choice functions on multi-dimensional restricted domains except for their study and Lavi, Mu'alem, and Nisan (2003). Mishra and Sen (2012) showed that when the domain of valuations are open intervals, then $\mathcal{G}^N$ is the class of all *neutral* and *strategy-proof* mechanisms. Lavi et al. (2003) showed that on order-based domains with conflicting preferences (e.g., domains for auctions), under additional restrictions on the social choice function, every *strategy-proof* mechanism must be a *Weighted-Groves* mechanism when the valuations of agents are bounded below.

The domains studied in Mishra and Sen (2012), or Lavi et al. (2003) do not apply in the task allocation problem we study.[13] Thus, even though the mechanisms in $\mathcal{G}$ are *strategy-proof* in our model, it remains an open question what other classes of *strategy-proof* mechanisms with well-specified functional forms exist in our model. For the case of restricted domains like ours, Lavi et al. (2003) give examples of *strategy-proof* mechanisms that are not *Weighted-Groves* mechanisms. However, they state that these examples are purely technical and economically non-interesting examples. Hence, class $\mathcal{G}$ still appears to be the most relevant class of *strategy-proof* mechanisms in our model. Our next result shows that, under $T-$*bounded-deficit*, there is no *neutral Weighted-Groves* mechanism which Pareto-dominates the mechanisms in $\mathcal{E}^{\gamma^T}$. We also show that a weak restriction on the weights guarantee that the mechanisms in $\mathcal{E}^{\gamma^T}$ are Pareto-undominated by any (not necessarily *neutral) Weighted-Groves* mechanism satisfying this restriction and $T-$*bounded-deficit*.

Theorem 4. *Let $T \in \mathbb{R}$.*

a) *The mechanisms in $\mathcal{E}^{\gamma^T}$ are Pareto-undominated by any* neutral *Weighted-Groves mechanism $G^s \in \mathcal{G}^N$ that satisfies $T-$bounded-deficit.*

b) *The mechanisms in $\mathcal{E}^{\gamma^T}$ are Pareto-undominated by any Weighted-Groves mechanism $G^s \in \mathcal{G}$ that satisfies $T-$bounded-deficit and is associated with $\omega \in \mathcal{W}$ such that there is $N = \{i, j\} \in \mathcal{N}$ with $\omega_i^N > 2\omega_j^N > 0$.*

3.4 Other Properties of the Egalitarian Mechanisms

If tasks are imposed on agents as in government requisitions and eminent domain, then agents do not have the option of refusing their task assignments, even if they may experience a negative utility. Also, if agents are collectively responsible for the completion of tasks (as in NIMBY problems), then they are collectively responsible for bearing the costs of the tasks and it may be natural that they should end up with negative utilities. In such cases, for each $\gamma \in \Gamma$, the mechanisms in $\mathcal{E}^\gamma$ are very appealing since they respect the following welfare lower bounds that are analogous to weak social participation constraints: for each $N \in \mathcal{N}$, no agent is worse off than the case where she is assigned all the tasks and compensated by an amount of money $\gamma(N)$. Let $\gamma \in \Gamma$.

$\gamma-$**Compensation Lower-Bound ($\gamma-$CLB):** For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$,

$$u(\varphi_i(c); c_i) \geq -c_i(\mathbb{A}) + \gamma(N).$$

Note that when $\gamma(N) = 0$ for each $N \in \mathcal{N}$, then under $\gamma-$*CLB*, no agent receives less than her *stand-alone utility* where she is assigned all the tasks and receives no compensation for it. The

[13] Section 6.1.1 in Mitra and Sen (2012) explains why their domain does not cover the domains for auctions/allocation of objects. In Lavi et al. (2003), conflicting preferences require that for each agent, her most preferred alternative must be least preferred by all other agents. This property holds for the problem of allocating desirable objects, but not for allocating tasks.

mechanisms in $\mathcal{E}^{\gamma^T}$ are the only Groves mechanisms satisfying $T-$*bounded-deficit* and γ^T-*CLB* where each agent is compensated an equal share of the upper bound on deficit, T.

On the other hand, if one insists on that no agent should experience a negative utility level (*individual rationality*), then the mechanisms in $\mathcal{E}$ do not satisfy this property unless there is an upper bound on the cost that any agent may incur. Suppose there exists $K \in \mathbb{R}_+$ such that for each $i \in \mathbb{N}$ and each $A \in 2^{\mathbb{A}}$, $c_i(A) \leq K$. Consider the domain of cost profiles comprised of such cost functions. Lemma 1 still holds on this domain. Hence, on this domain, a mechanism is *assignment-efficient, strategy-proof, welfare-egalitarian,* and *individually rational* if and only if it belongs to $\mathcal{E}^\gamma$ where $\gamma : \mathcal{N} \to \mathbb{R}$ is such that for each $N \in \mathcal{N}$, $\gamma(N) \geq K$. To see this, by *individual rationality,* for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $u(E_i^{\gamma,\tau}(c); c_i) \geq 0$, that is, by equation (4), $\gamma(N) \geq W(c)$. Since this is true for each $c \in \mathcal{C}^N$, and $\gamma(N)$ is independent of c, we have $\gamma(N) \geq \max_{c \in \mathcal{C}^N}\{W(c)\} = K$.

Yengin (2012a,b) provide alternative characterizations of the class of *Egalitarian* mechanisms. These characterizations, together with our results in this paper indicate that several equity axioms that are incompatible in many other models can be satisfied jointly in our multi-object-demand model. However, the price of the compatibility is to overwrite any welfare differentials. In other words, for Groves mechanisms, joint implication of most fairness axioms is *welfare-egalitarianism* as displayed in Table 1.

	On the subadditive domain:		
	Theorem 1:		
(i)	if $\lvert N\rvert \geq 3$, *egalitarian-equivalence+no-envy*	$\Leftrightarrow$	$\mathcal{E}$
	Theorem 2:		
(ii)	*egalitarian-equivalence + no-envy+population monotonicity*	$\Rightarrow$	$\mathcal{E}$
	Proposition 3b:		
(iii)	*egalitarian-equivalence + no-envy+*$T-$ *bounded-deficit*	$\Leftrightarrow$	$\mathcal{E}^{T-BD}$
	Yengin (2012a):		
(iv)	*no-envy+solidarity*	$\Leftrightarrow$	$\mathcal{E}$
	On any domain:		
	Lemma 1:		
(v)	*welfare-egalitarianism*	$\Leftrightarrow$	$\mathcal{E}$
	Yengin (2012b):		
(vi)	*egalitarian-equivalence +* $T-$ *bounded-deficit*	$\Leftrightarrow$	$\mathcal{E}^{T-BD}$
	Yengin (2012a):		
(vii)	*egalitarian-equivalence+solidarity*	$\Leftrightarrow$	$\mathcal{E}$
(viii)	*order-preservation + solidarity*	$\Leftrightarrow$	$\mathcal{E}$
(ix)	minimum deficit among all Groves mechanisms satisfying γ-*CLB*	$\Leftrightarrow$	$\mathcal{E}^\gamma$
(x)[14]	$m^{k,T}$ *− bounded-deficit +* γ^T-*CLB*	$\Leftrightarrow$	$\mathcal{E}^{\gamma^T}$

Table 1: Results under *assignment-efficiency* and *strategy-proofness.*

Note that for each $T \in \mathbb{R}$, the *Egalitarian* mechanisms in $\mathcal{E}^{\gamma^T}$ satisfy all the axioms mentioned in Table 1. Hence, they appear to be the best candidates to satisfy several different equity and solidarity requirements as well as generating bounded deficits and respecting certain welfare bounds. Moreover, under $T-$*bounded-deficit,* $\mathcal{E}^{\gamma^T}$ is either the Pareto-dominant or the Pareto-undominated

[14]Taking $T = 0$, a Groves mechanism generates *no deficit* (or a deficit up to $kW(c)$ with $k \geq -(n-1)$) and guarantees each agent at least her stand-alone utility if and only if it is an *Egalitarian* mechanism $E^{\gamma^T,\tau}$ such that for each $N \in \mathcal{N}$, $\gamma^T(N) = 0$.

class among several subclasses of Groves mechanisms as stated in Theorem 3. They are also Pareto-undominated by any Groves mechanism or *neutral Weighted-Groves* mechanism satisfying $T-$*bounded-deficit* (Theorem 4). These results reinforce the importance of the class $\mathcal{E}^{\gamma^T}$ in the economic setting we study. Note that the Pivotal/Vickrey mechanisms, which have been the focus of most of the literature on the Groves mechanisms, violate *egalitarian-equivalence, population monotonicity, solidarity*, $m^{k,T}-$*bounded-deficit*, and *no-deficit*.

4 Conclusion

We have characterized the class of *envy-free* and *egalitarian-equivalent* Groves mechanisms in a multi-object-demand model where the joint implications of these two fairness axioms have not been studied before. Our results clarified the differences one would get and the factors that derive those differences on the compatibility of these two axioms in the single-object-demand model versus in our model.

By imposing an upper bound on deficit, $T \in \mathbb{R}$, we singled-out $\mathcal{E}^{\gamma^T}$ as the Pareto-dominant class among all *envy-free* and *egalitarian-equivalent* Groves mechanisms on the *subadditive* domain. In Theorem 3, we showed that this Pareto-dominant class is actually still Pareto-dominant even if (i) we drop *no-envy*, or (ii) replace *egalitarian-equivalence* with *solidarity*, or(iii) replace *no-envy* and *egalitarian-equivalence* with *solidarity* and *unanimity/order preservation*. Also, under $T-$*bounded-deficit*, no Groves mechanism and no *neutral Weighted-Groves* mechanism can Pareto-dominate an *Egalitarian* mechanism in $\mathcal{E}^{\gamma^T}$.

In our working paper version of this paper (Yengin, 2011), we have related *no-envy* and *egalitarian-equivalence* to the *"equality of what"* debate in the political philosophy of distributive justice and built a link between resource and opportunity egalitarianism, and welfare-egalitarianism. Specifically, we proposed that, by By Theorem 1, on the subadditive domain and when there are at least three agents, under *assignment-efficiency* and *strategy-proofness,* the two axioms that support *equality of resources* and *equality of opportunities,* namely, *egalitarian-equivalence* and *no-envy*, together imply *equality of welfares.* (see Yengin, 2011).

In the multi-object demand case, *no-envy* does not imply *assignment-efficiency*. Whether or not this result changes if we additionally impose *strategy-proofness* is an open and non-trivial question. In particular, characterization of *strategy-proof* mechanisms satisfying *no-envy* and/or *egalitarian-equivalence* in the multi-object-demand model is an open question left for future research.

5 Appendix

Proof of Theorem 1:

If Part: Pick an *assignment-efficient* and *strategy-proof* mechanism. By Lemma 2, it is a Groves mechanism $G^{h,\tau}$ for some $h \in \mathcal{H}$ and $\tau \in \mathcal{T}$. Let $G^{h,\tau} \in \widehat{\mathcal{E}}$. Then, there is $\gamma : \mathcal{N} \to \mathbb{R}$ such that $G^{h,\tau} = \widehat{E}^{\gamma,\tau}$ where (18), (19), and (20) hold.
We will show that $G^{h,\tau}$ is (a) *envy-free* on the *subadditive* domain, and (b) *egalitarian-equivalent* on every domain.

(a) Assume, by contradiction, that $G^{h,\tau}$ is not *envy-free* on the *subadditive* domain. Then, there are $N \in \mathcal{N}$, $c \in \mathcal{C}^N_{sub}$, and $\{i, j\} \subseteq N$ such that $u(G^{h,\tau}_i(c); c_i) < u(G^{h,\tau}_j(c); c_i)$. This inequality, (3), and (4) together imply

$$\begin{aligned} -W(c) + h_i(c_{-i}) &< -c_i(A^\tau_j(c)) + t^{h,\tau}_j(c), \\ &= -c_i(A^\tau_j(c)) - W(c) + c_j(A^\tau_j(c)) + h_j(c_{-j}). \end{aligned} \tag{26}$$

First, consider equations (18) and (19). By (26), $c_i(A_j^\tau(c)) < c_j(A_j^\tau(c))$. This inequality implies that $A_j^\tau(c) \neq \emptyset$ and since c is *subadditive,* we have

$$c_i(A_i^\tau(c) \cup A_j^\tau(c)) \leq c_i(A_i^\tau(c)) + c_i(A_j^\tau(c)) < c_i(A_i^\tau(c)) + c_j(A_j^\tau(c)).$$

Then, it would be less costly than $W(c)$, if i was assigned $(A_i^\tau(c) \cup A_j^\tau(c))$ and j was assigned no task, which contradicts that $A^\tau(c)$ is an *efficient assignment.*
Now, consider equation (20). By (26),

$$c_j(\mathbb{A}) < -c_i(A_j^\tau(c)) + c_j(A_j^\tau(c)) + c_i(\mathbb{A}). \tag{27}$$

By (27), $A_j^\tau(c) \neq \mathbb{A}$. Since c is *subadditive* and $\mathbb{A} = A_i^\tau(c) \cup A_j^\tau(c)$, then $c_i(\mathbb{A}) \leq c_i(A_i^\tau(c)) + c_i(A_j^\tau(c))$. This inequality and (27) together imply $c_j(\mathbb{A}) < c_i(A_i^\tau(c)) + c_j(A_j^\tau(c))$. Then, it would be less costly than $W(c)$, if j was assigned all the tasks, which contradicts that $A^\tau(c)$ is an *efficient assignment.*

(b) Now, we show that $G^{h,\tau}$ is *egalitarian-equivalent.* Let $N \in \mathcal{N}$ and $c \in \mathcal{C}^N$.
First, consider (18) and (19). By (4), for each $i \in N$, $u(G_i^{h,\tau}(c); c_i) = -W(c) + \gamma(N)$. Let $R(c) = \emptyset$ and $r(c) = -W(c) + \gamma(N)$. Then, for each $i \in N$, $u(G_i^{h,\tau}(c); c_i) = -c_i(R(c)) + r(c)$. Hence, $G^{h,\tau}$ is *egalitarian-equivalent.*
Next, consider (20). By (4), for each $i \in N$, $u(G_i^{h,\tau}(c); c_i) = -W(c) + \gamma(N) + c_j(\mathbb{A})$ for $j \in N\backslash\{i\}$. Let $R(c) = \mathbb{A}$ and $r(c) = -W(c) + \sum_{i \in N} c_i(\mathbb{A}) + \gamma(N)$. Then, for each $i \in N$, $u(G_i^{h,\tau}(c); c_i) = -c_i(R(c)) + r(c)$. Hence, $G^{h,\tau}$ is *egalitarian-equivalent.*

Only-if Part: Pick an *assignment-efficient* and *strategy-proof* mechanism. By Lemma 2, it is a Groves mechanism $G^{h,\tau}$ for some $h \in \mathcal{H}$ and $\tau \in \mathcal{T}$. Let $G^{h,\tau}$ be *egalitarian-equivalent* and *envy-free* on the *subadditive* domain.

By Theorem 1 in Yengin (2012a) (adapted to our variable population setting), if a Groves mechanism is *egalitarian-equivalent*, then for economies with different populations, different reference set of tasks can be chosen; but for economies with the same population N, the same reference set of tasks $\overline{R}(N)$ must work. Moreover, by Yengin (2012a), a Groves mechanism is *egalitarian-equivalent* if and only if for each $N \in \mathcal{N}$, there are a real number $\gamma(N) \in \mathbb{R}$ and a reference set $\overline{R}(N) \in 2^{\mathbb{A}}$ such that for each $i \in N$,

$$h_i(c_{-i}) = \gamma(N) + \sum_{j \in N\backslash\{i\}} c_j(\overline{R}(N)) \text{ for each } c \in \mathcal{C}^N. \tag{28}$$

By Theorem 1 in Pápai (2003) (adapted to our variable-population and undesirable-objects setting), if $G^{h,\tau}$ is *envy-free* on the *subadditive* domain, then there is a list of functions indexed by populations, $\{\sigma_N\}_{N \in \mathcal{N}}$ with $\sigma_N : \mathbb{R}_+ \to \mathbb{R}$ such that for each $N \in \mathcal{N}$, each $i \in N$, and each $c \in \mathcal{C}_{sub}^N$,

$$h_i(c_{-i}) = \sigma_N(W(c_{-i})). \tag{29}$$

By (28) and (29), for each $N \in \mathcal{N}$ and each pair $\{i, j\} \subseteq N$,

$$\sigma_N(W(c_{-i})) - \sigma_N(W(c_{-j})) = c_j(\overline{R}(N)) - c_i(\overline{R}(N)) \text{ for each } c \in \mathcal{C}_{sub}^N. \tag{30}$$

Using (28), (29), and, (30), we will prove that $G^{h,\tau} = \widehat{E}^{\gamma,\tau}$ by showing, on the *subadditive* domain, the equivalence of equation (28) to (18) when $|N| > 2$; and (28) to (19) or (20) when $|N| = 2$. To achieve this, we need to prove the following two cases:

Case 1: For each $N \in \mathcal{N}$ with $|N| > 2$, there is $\gamma(N) \in \mathbb{R}$ such that (28) holds for $\overline{R}(N) = \emptyset$ for each $c \in \mathcal{C}^N_{sub}$. That is, equality (28) is equivalent to (18).

Proof of Case 1:

Let $N \in \mathcal{N}$ with $|N| > 2$. By *egalitarian-equivalence,* there are $\gamma(N) \in \mathbb{R}$ and $\overline{R}(N) \in 2^{\mathbb{A}}$ such that (28) holds for each $c \in \mathcal{C}^N_{sub}$.

Claim: For each $c \in \mathcal{C}^N_{sub}$ and each pair $\{i, j\} \subset N$, $c_i(\overline{R}(N)) = c_j(\overline{R}(N))$.

Note that the Claim holds for each $c \in \mathcal{C}^N_{sub}$; and by (28), $\overline{R}(N)$ is same for each $c \in \mathcal{C}^N_{sub}$. This is possible if and only if $\overline{R}(N) = \emptyset$. This would prove that Case 1 holds.

Now, we will show that the Claim is true:
Assume, by contradiction to the claim, that there is $c \in \mathcal{C}^N_{sub}$ such that for some pair $\{i, j\} \subset N$,

$$c_i(\overline{R}(N)) \neq c_j(\overline{R}(N)). \tag{31}$$

Without loss of generality, let $W(c_{-j}) \leq W(c_{-i})$. Let $\Delta \equiv c_j(\overline{R}(N)) - c_i(\overline{R}(N))$. Let $\widehat{c} \in \mathcal{C}^N_{ad}$ be as follows:

(i) for each $A \in 2^{\mathbb{A}}$, $\widehat{c}_i(A) = \frac{|A|\ W(c_{-j})}{|\mathbb{A}|}$,

(ii) there is $\varepsilon > \max\{-\Delta, 0\}$ such that for each $A \in 2^{\mathbb{A}}$, $\widehat{c}_j(A) = |A| \left(\frac{W(c_{-i})}{|\mathbb{A}|} + \Delta + \varepsilon \right)$,

(iii) for each $k \in N \backslash \{i, j\}$ and each $A \in 2^{\mathbb{A}}$, $\widehat{c}_k(A) = \frac{|A|\ W(c_{-i})}{|\mathbb{A}|}$.

Note that $\Delta + \varepsilon > \max\{\Delta, 0\}$. Hence, for each $A \in (2^{\mathbb{A}} \backslash \emptyset)$, $\widehat{c}_j(A) > \widehat{c}_k(A)$.
By (i) and (ii), for each $A \in (2^{\mathbb{A}} \backslash \emptyset)$,

$$\begin{aligned} \widehat{c}_j(A) - \widehat{c}_i(A) &= |A| \left(\frac{W(c_{-i}) - W(c_{-j})}{|\mathbb{A}|} + \Delta + \varepsilon \right), \\ &> \Delta. \end{aligned} \tag{32}$$

By (ii) and (iii), (I) $W(\widehat{c}_{-i}) = \widehat{c}_k(\mathbb{A}) = W(c_{-i})$ for some $k \in N \backslash \{i, j\}$.
Since $W(c_{-j}) \leq W(c_{-i})$, by (i) and (iii), (II) $W(\widehat{c}_{-j}) = \widehat{c}_i(\mathbb{A}) = W(c_{-j})$.
By (I), (II), and (30), $c_j(\overline{R}(N)) - c_i(\overline{R}(N)) = \Delta = \widehat{c}_j(\overline{R}(N)) - \widehat{c}_i(\overline{R}(N))$. This equality and (32) together imply $\overline{R}(N) = \emptyset$. This implies $c_j(\overline{R}(N)) = c_i(\overline{R}(N))$, which contradicts (31). Hence, Claim must be true.

Case 2: For each $N \in \mathcal{N}$ with $|N| = 2$, there is $\gamma(N) \in \mathbb{R}$ such that (28) holds either for $\overline{R}(N) = \emptyset$ for each $c \in \mathcal{C}^N_{sub}$; or for $\overline{R}(N) = \mathbb{A}$ for each $c \in \mathcal{C}^N_{sub}$. That is, equality (28) is equivalent to (19) or (20).

Proof of Case 2:

Let $N \in \mathcal{N}$ with $|N| = 2$. Without loss of generality, let $N = \{i, j\}$. By *egalitarian-equivalence,* there are $\gamma(N) \in \mathbb{R}$ and $\overline{R}(N) \in 2^{\mathbb{A}}$ such that (28) holds for each $c \in \mathcal{C}^N_{sub}$. We will show that $\overline{R}(N) \in \{\emptyset, \mathbb{A}\}$.

Let $c \in \mathcal{C}^N_{sub}$ and $\widehat{c} = (c_i, \widehat{c}_j) \in \mathcal{C}^N_{sub}$ be such that (I) $\widehat{c}_j(\mathbb{A}) = c_j(\mathbb{A})$ and (II) for each $A \in (2^{\mathbb{A}} \backslash \{\emptyset, \mathbb{A}\})$, $\widehat{c}_j(A) \neq c_j(A)$. By (I), $W(\widehat{c}_{-i}) = c_j(\mathbb{A}) = W(c_{-i})$. This equality and (29) together imply $h_i(\widehat{c}_{-i}) = h_i(c_{-i})$. This equality and (28) together imply $\widehat{c}_j(\overline{R}(N)) = c_j(\overline{R}(N))$. This equality and (II) together imply that $\overline{R}(N) \in \{\emptyset, \mathbb{A}\}$. This proves Case 2. ∎

Proof of Theorem 2: Let $E^{\gamma,\tau}$ be an *Egalitarian mechanism* such that γ is as in (23). By Theorem 1, $E^{\gamma,\tau}$ is an *egalitarian-equivalent* Groves mechanism that is *envy-free* on the *subadditive* domain. By Lemma 4, $E^{\gamma,\tau}$ is *population monotonic.*

Conversely, let $G^{h,\tau}$ be a Groves mechanism that is *egalitarian-equivalent, population monotonic*, and *envy-free* on the *subadditive* domain. By Lemma 4, $h \in \mathcal{H}$ is as in (21). By Theorem 1, $G^{h,\tau}$ belongs to $\widehat{\mathcal{E}}$. Hence, there is $\gamma \in \Gamma$ such that $G^{h,\tau} = \widehat{E}^{\gamma,\tau}$.
Assume that $G^{h,\tau} \in (\widehat{\mathcal{E}} \backslash \mathcal{E})$. Then, by (20), there is $N' = \{i,j\} \in \mathcal{N}$ such that $h_i(c_j) = c_j(\mathbb{A}) + \gamma(N')$ for each $(c_i, c_j) \in \mathcal{C}^{N'}_{sub}$.
Let $N = \{i,j,k\} \in \mathcal{N}$. Then, by (18), $h_i(c_j, c_k) = \gamma(N)$ for each $c = (c_i, c_j, c_k) \in \mathcal{C}^N_{sub}$.
Let $\varepsilon > \max\{0, \gamma(N') - \gamma(N)\}$. Let $\widehat{c}_j \in \mathcal{C}_{ad}$ be such that for each $A \in (2^{\mathbb{A}} \backslash \{\emptyset\})$,

$$\widehat{c}_j(A) = \frac{|A|}{|\mathbb{A}|}\left(\gamma(N) - \gamma(N') + \varepsilon\right). \tag{33}$$

Note that $(c_i, \widehat{c}_j) \in \mathcal{C}^{N'}_{sub}$ and $(c_i, \widehat{c}_j, c_k) \in \mathcal{C}^N_{sub}$. By (20), (I) $h_i(\widehat{c}_j) = \widehat{c}_j(\mathbb{A}) + \gamma(N')$. By (18), (II) $h_i(\widehat{c}_j, c_k) = \gamma(N)$.
By (33), $\widehat{c}_j(\mathbb{A}) = \gamma(N) - \gamma(N') + \varepsilon$. This equality, (I), and (II) together imply $h_i(\widehat{c}_j) = \gamma(N) + \varepsilon > h_i(\widehat{c}_j, c_k)$. This inequality contradicts (21). Hence, $G^{h,\tau} \notin (\widehat{\mathcal{E}} \backslash \mathcal{E})$.
Let $G^{h,\tau} \in \mathcal{E}$. Then, there is $\gamma : \mathcal{N} \rightarrow \mathbb{R}$ such that $G^{h,\tau} = E^{\gamma,\tau}$.
Note that for each pair $\{N, N'\} \subset \mathcal{N}$ such that $N' \subset N$, each $i \in N'$, and each $c \in \mathcal{C}^N$, $h_i(c_{-i}) = \gamma(N)$ and $h_i(c_{N' \backslash \{i\}}) = \gamma(N')$. By (21), $\gamma(N) \geq \gamma(N')$. Hence, $E^{\gamma,\tau} \in \mathcal{E}^{\gamma}$ where $\gamma : \mathcal{N} \rightarrow \mathbb{R}$ is as in (23). ■

Proof of Theorem 3: Let $T \in \mathbb{R}$.

a) Let $\widehat{E}^{\gamma,\tau} \in \widehat{\mathcal{E}}^{T-BD} \backslash \mathcal{E}^{\gamma^T}$. Then, by Lemma 5 and Proposition 3a, for each $N \in \mathcal{N}$, $\gamma(N) < T/n$. By (4), for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$, $u(\widehat{E}^{\gamma,\tau}_i(c); c_i) = -W(c) + \gamma(N) < -W(c) + T/n$. For each $E^{\gamma^T,\tau} \in \mathcal{E}^{\gamma^T}$, since for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$, $u(E^{\gamma^T,\tau}_i(c); c_i) = -W(c) + T/n$, $E^{\gamma^T,\tau}$ Pareto-dominates $\widehat{E}^{\gamma,\tau}$.

b) By Proposition 2 in Yengin (2012b), an *egalitarian-equivalent* Groves mechanism $G^{h,\tau}$ satisfies $T-$*bounded-deficit* if and only if $G^{h,\tau} \in \mathcal{E}^{T-BD}$. Let $G^{h,\tau} \notin \mathcal{E}^{\gamma^T}$ be such a mechanism. That is, $G^{h,\tau} \in \mathcal{E}^{T-BD} \backslash \mathcal{E}^{\gamma^T}$. By Proposition 3a, $G^{h,\tau} \in \widehat{\mathcal{E}}^{T-BD} \backslash \mathcal{E}^{\gamma^T}$. By part (a), for each $E^{\gamma^T,\tau} \in \mathcal{E}^{\gamma^T}$, $E^{\gamma^T,\tau}$ Pareto-dominates $G^{h,\tau}$.

Let $G^{h,\tau}$ generate the minimal surplus among all Groves mechanisms that satisfy *egalitarian-equivalence* and $T-$*bounded-deficit*. For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, the budget surplus generated by $G^{h,\tau}$ is $-\sum\limits_{i \in N} t^{h,\tau}_i(c) = (n-1)W(c) - \sum\limits_{i \in N} h_i(c_{-i})$. Hence, the surplus is minimal if and only if $\sum\limits_{i \in N} h_i(c_{-i})$ is maximal. Since $G^{h,\tau} \in \mathcal{E}^{T-BD}$, $\sum\limits_{i \in N} h_i(c_{-i}) = n\gamma(N) \leq T$ is maximal if and only if for each $N \in \mathcal{N}$, $\gamma(N) = \frac{T}{|N|}$. Hence, $G^{h,\tau} \in \mathcal{E}^{\gamma^T}$.

c) By Corollary 1 in Yengin (2012a), on the *subadditive* domain, a Groves mechanism $G^{h,\tau}$ satisfies *no-envy* and *solidarity* if and only if $G^{h,\tau} \in \mathcal{E}$. Hence, on this domain, a Groves mechanism $G^{h,\tau}$ satisfies *no-envy, solidarity,* and $T-$*bounded-deficit* if and only if $G^{h,\tau} \in \mathcal{E}^{T-BD}$. The rest of the proof is as in part (b).

d) By Proposition 2b in Yengin (2012a), a Groves mechanism $G^{h,\tau}$ satisfies *order preservation* and *solidarity* if and only if $G^{h,\tau} \in \mathcal{E}$. The rest of the proof is as in part (c).

e) Follows from Proposition 4b in Yengin (2012a).

f) Let $E^{\gamma^T,\tau} \in \mathcal{E}^{\gamma^T}$. Assume, by contradiction, there is a Groves mechanism $G^{h,\tau}$ which satisfies $T-$*bounded-deficit* and Pareto-dominates $E^{\gamma^T,\tau}$. By (4), for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$, $u(G_i^{h,\tau}(c); c_i) = -W(c) + h_i(c_{-i})$ and $u(E_i^{\gamma^T,\tau}(c); c_i) = -W(c) + \frac{T}{n}$. Since $G^{h,\tau}$ Pareto-dominates $E^{\gamma^T,\tau}$, there is $N \in \mathcal{N}$, $c \in \mathcal{C}^N$, and $j \in N$ such that $h_j(c_{-j}) > \frac{T}{n}$. Let $c_j^0 \in \mathcal{C}$ be such that $c_j^0(\mathbb{A}) = 0$ and $c' = (c_j^0, c_{-j}) \in \mathcal{C}^N$. Since $c'_{-j} = c_{-j}$, $h_j(c'_{-j}) = h_j(c_{-j}) > \frac{T}{n}$. Since $W(c') = c_j^0(\mathbb{A}) = 0$, then $\sum_{i\in N} t_i^{h,\tau}(c') = -(n-1)W(c') + \sum_{i\in N} h_i(c'_{-i}) = \sum_{i\in N\backslash\{j\}} h_i(c'_{-i}) + h_j(c_{-j})$. Since $G^{h,\tau}$ Pareto-dominates $E^{\gamma^T,\tau}$, for each $i \in N\backslash\{j\}$, $h_i(c'_{-i}) \geq \frac{T}{n}$. Then, $\sum_{i\in N} t_i^{h,\tau}(c') > T$, which contradicts $T-$*bounded-deficit*. This completes the proof. ■

Proof of Theorem 4: Let $T \in \mathbb{R}$ and $E^{\gamma^T,\tau} \in \mathcal{E}^{\gamma^T}$.

a) Assume, by contradiction, that there is a *neutral Weighted-Groves* mechanism $G^s \in \mathcal{G}^N$ that satisfies T-*bounded-deficit* and Pareto-dominates $E^{\gamma^T,\tau}$. For each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $N' \subseteq N$, let

$$X^{\omega^N}(c_{N'}) = \min\{\sum_{j\in N'} \omega_j^N c_j(A'_j) : (A'_j)_{j\in N'} \in \mathcal{A}(N')\}.$$

Note that for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $X^{\omega^N}(c) = \sum_{j\in N} \omega_j^N c_j(A_j^s(c))$. Then, by (25), for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$ with $\omega_i^N > 0$,

$$t_i^s(c) = -\frac{1}{\omega_i^N}\left[\sum_{j\in N\backslash\{i\}} \omega_j^N c_j(A_j^s(c))\right] + h_i(c_{-i}) \quad (34)$$

$$= c_i(A_i^s(c)) - \frac{1}{\omega_i^N} X^{\omega^N}(c) + h_i(c_{-i}), \quad (35)$$

which implies that

$$u(A_i^s(c), t_i^s(c); c_i) = -\frac{1}{\omega_i^N} X^{\omega^N}(c) + h_i(c_{-i}). \quad (36)$$

Since G^s Pareto-dominates $E^{\gamma^T,\tau}$, for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$ with $\omega_i^N > 0$, by (36), $u(G_i^s(c); c_i) = -\frac{1}{\omega_i^N} X^{\omega^N}(c) + h_i(c_{-i}) \geq u(E_i^{\gamma^T,\tau}(c); c_i) = -W(c) + \frac{T}{n}$, that is,

$$h_i(c_{-i}) \geq \frac{1}{\omega_i^N} X^{\omega^N}(c) - W(c) + \frac{T}{n}. \quad (37)$$

We have two possible cases:

Case I: There is $N \in \mathcal{N}$ and $i \in N$ such that $\omega_i^N = 0$.

Case II: For each $N \in \mathcal{N}$ and each $i \in N$, $\omega_i^N > 0$.

Case I: Let $N \in \mathcal{N}$ be such that there is a pair $\{i^*, j^*\} \subseteq N$ where $\omega_{i^*}^N > 0$ and $\omega_{j^*}^N = 0$. Let $N^0 \equiv \{j \in N : \omega_j^N = 0\}$ and $N^+ = N\backslash N^0$. Note that $j^* \in N^0$ and $i^* \in N^+$. Let $n^+ = |N^+|$ and $n^0 = |N^0|$.

Let $\varepsilon > 0$ and $c \in \mathcal{C}_{ad}^N$ be such that

(i) for each $A \in (2^{\mathbb{A}}\backslash\emptyset)$, $c_{i^*}(A) = |A|\varepsilon$, and

(ii) for each $i \in N\backslash\{i^*\}$ and each $A \in (2^{\mathbb{A}}\backslash\emptyset)$, $c_i(A) > nc_{i^*}(\mathbb{A})$.

Note that by (i) and (ii), $W(c) = |\mathbb{A}|\varepsilon$ and for each $i \in N$,

$$u(E_i^{\gamma^T,\tau}(c); c_i) = -W(c) + \frac{T}{n} = -c_{i^*}(\mathbb{A}) + \frac{T}{n}. \tag{38}$$

Note also that for each $A' \in \mathcal{A}(N)$, $\sum_{j\in N} \omega_j^N c_j(A'_j) = \sum_{j\in N^+} \omega_j^N c_j(A'_j)$. By (24), $A^s(c) \in \arg\min\{\sum_{j\in N^+} \omega_j^N c_j(A'_j) : (A'_j)_{j\in N} \in \mathcal{A}(N)\}$. Since for each $i \in N^+$ and each $A \in (2^{\mathbb{A}}\backslash\emptyset)$, $c_i(A) > 0$, then for each $i \in N^+$, $A_i^s(c) = \emptyset$. That is, $\sum_{j\in N^+} \omega_j^N c_j(A'_j)$ is minimized at an assignment $A' \in \mathcal{A}(N)$ which allocates the tasks only to the agents in N^0. Thus, by (35) and (36), for each $i \in N^+$,

$$u(A_i^s(c), t_i^s(c); c_i) = t_i^s(c) = h_i(c_{-i}). \tag{39}$$

Claim I: For each $i \in N^+$, $h_i(c_{-i}) \geq \frac{T}{n}$.
Proof of Claim I: Let $i \in N^+$ and $c_i^0 \in \mathcal{C}_{ad}$ be such that for each $A \in (2^{\mathbb{A}}\backslash\emptyset)$, $c_i^0(A) = 0$. Let $c' = (c_i^0, c_{-i}) \in \mathcal{C}_{ad}^N$. Note that $X^{\omega^N}(c') = W(c') = 0$. By (37), $h_i(c'_{-i}) \geq \frac{T}{n}$. Since $c'_{-i} = c_{-i}$, then $h_i(c_{-i}) \geq \frac{T}{n}$. ◊

Claim II: There is $k \in N^0$ such that $-c_k(A_k^s(c)) + t_k^s(c) < -c_{i^*}(\mathbb{A}) + \frac{T}{n}$.
Proof of Claim II: Suppose not. Then, for each $i \in N^0$, $-c_i(A_i^s(c)) + t_i^s(c) \geq -c_{i^*}(\mathbb{A}) + \frac{T}{n}$. Then,

$$n^0 c_{i^*}(\mathbb{A}) - \frac{n^0 T}{n} + \sum_{i\in N^0} t_i^s(c) \geq \sum_{i\in N^0} c_i(A_i^s(c)). \tag{40}$$

Note that since for each $i \in N^+$, $A_i^s(c) = \emptyset$, then there is $j \in N^0$ such that $A_j^s(c) \neq \emptyset$. This fact together with (ii) implies that $\sum_{i\in N^0} c_i(A_i^s(c)) > n^0 c_{i^*}(\mathbb{A})$. This inequality together with (40) imply that $\sum_{i\in N^0} t_i^s(c) > \frac{n^0 T}{n}$. Note that by Claim I and (39), $\sum_{i\in N^+} t_i^s(c) \geq \frac{n^+ T}{n}$. All together, $\sum_{i\in N} t_i^s(c) > T$ which contradicts $T-$*bounded-deficit.* Thus, Claim II must be true. ◊

Since $u(G_k^s(c); c_k) = -c_k(A_k^s(c)) + t_k^s(c)$, by (38) and Claim II, $u(G_k^s(c); c_k) < u(E_k^{\gamma^T,\tau}(c); c_k)$. This contradicts that G^s Pareto-dominates $E^{\gamma^T,\tau}$. This completes the proof for Case I. ♦

Case II: There are two possible subcases:
Case IIa: For each $N \in \mathcal{N}$, $\omega_i^N = \omega_j^N$ for each pair $\{i, j\} \subseteq N$. In this case, G^s is a Groves mechanism, which, by Theorem 3f, contradicts that G^s Pareto-dominates $E^{\gamma^T,\tau}$. To see that G^s is a Groves mechanism, for each $N \in \mathcal{N}$, let $\overline{w}(N) > 0$ be such that for each $i \in N$, $\omega_i^N = \overline{w}(N)$. Then, for each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, $A^s(c) = \arg\min\{\overline{w}(N) \sum_{j\in N} c_j(A'_j) : (A'_j)_{j\in N} \in \mathcal{A}(N)\}$; hence, $A^s(c) = A^\tau(c)$ and by (34), $t_i^s(c) = t_i^{h,\tau}(c)$ for each $i \in N$. Then, for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$, $u(G_i^s(c); c_i) = u(G_i^{h,\tau}(c); c_i)$.

Case IIb: There is $N \in \mathcal{N}$ and a pair $\{i', j'\} \subseteq N$ such that $0 < \omega_{j'}^N < \omega_{i'}^N$. Let $N^{min} \equiv \{j \in N : 0 < \omega_j^N \leq \omega_i^N$ for each $i \in N\backslash\{j\}\}$ and $\widehat{N} \equiv N\backslash N^{min}$. Note that $i' \in \widehat{N}$, hence, $\widehat{N} \neq \emptyset$.

Let $c \in \mathcal{C}^N_{ad}$ be such that for each $i \in N$ and each $A \in (2^{\mathbb{A}} \backslash \emptyset)$, $c_i(A) > 0$, and for each $i \in N^{min}$ and each $A \in (2^{\mathbb{A}} \backslash \emptyset)$, $c_i(A) > \max\{\frac{1}{\omega_i^N} X^{\omega^N}(c_{\widehat{N}}), W(c_{\widehat{N}})\}$. Note that $X^{\omega^N}(c) = X^{\omega^N}(c_{\widehat{N}})$ and $W(c) = W(c_{\widehat{N}})$.
Let $k \in N^{min}$ and $c^0_k \in \mathcal{C}_{ad}$ be such that for each $A \in (2^{\mathbb{A}} \backslash \emptyset)$, $c^0_k(A) = 0$. Let $c' \in \mathcal{C}^N_{ad}$ be such that $c' = (c^0_k, c_{-k})$. Note that $A^\tau_k(c') = A^s_k(c') = \mathbb{A}$ and $W(c') = 0 = X^{\omega^N}(c')$. Hence, by (37), for each $i \in N$, $h_i(c'_{-i}) \geq \frac{T}{n}$. This inequality and (35) together imply that, for each $i \in N$,

$$t^s_i(c') = h_i(c'_{-i}) \geq \frac{T}{n}. \tag{41}$$

Since $c'_{-k} = c_{-k}$, by (37),

$$t^s_k(c') = h_k(c_{-k}) \geq \frac{1}{\omega^N_k} X^{\omega^N}(c) - W(c) + \frac{T}{n}. \tag{42}$$

By (41) and (42),

$$\begin{aligned} \sum_{i \in N} t^s_i(c') &= t^s_k(c') + \sum_{i \in N \backslash \{k\}} t^s_i(c') \\ &\geq T + \frac{1}{\omega^N_k} X^{\omega^N}(c) - W(c). \end{aligned} \tag{43}$$

By $T-$*bounded-deficit*, $T \geq \sum_{i \in N} t^s_i(c')$. This inequality and (43) together imply

$$\begin{aligned} \omega^N_k W(c) &\geq X^{\omega^N}(c) \\ \omega^N_k W(c_{\widehat{N}}) &\geq X^{\omega^N}(c_{\widehat{N}}) \\ \omega^N_k \min\{\sum_{i \in \widehat{N}} c_i(A'_i) &: (A'_i)_{i \in \widehat{N}} \in \mathcal{A}(\widehat{N})\} \geq \min\{\sum_{i \in \widehat{N}} \omega^N_i c_i(A'_i) : (A'_i)_{i \in \widehat{N}} \in \mathcal{A}(\widehat{N})\} \\ \min\{\sum_{i \in \widehat{N}} \omega^N_k c_i(A'_i) &: (A'_i)_{i \in \widehat{N}} \in \mathcal{A}(\widehat{N})\} \geq \min\{\sum_{i \in \widehat{N}} \omega^N_i c_i(A'_i) : (A'_i)_{i \in \widehat{N}} \in \mathcal{A}(\widehat{N})\}, \end{aligned}$$

which contradicts that for each $i \in \widehat{N}$ and each $A \in (2^{\mathbb{A}} \backslash \emptyset)$, $c_i(A) > 0$ and $\omega^N_i > \omega^N_k$. This completes the proof for Case II. ♦

b) Assume, by contradiction, that there is a *Weighted-Groves* mechanism $G^s \in \mathcal{G}$ that satisfies *T-bounded-deficit*, Pareto-dominates $E^{\gamma^T, \tau}$, and is associated with $\omega \in \mathcal{W}$ such that there is $N = \{i, j\} \in \mathcal{N}$ with $\omega^N_i > 2\omega^N_j > 0$. For each $N \in \mathcal{N}$ and each $c \in \mathcal{C}^N$, let

$$\begin{aligned} Z^{\omega^N}(c) &= \min\{\sum_{j \in N} \omega^N_j c_j(A'_j) + F(A') : A' = (A'_j)_{j \in N} \in \mathcal{A}(N)\} \\ &= \sum_{j \in N} \omega^N_j c_j(A^s_j(c)) + F(A^s(c)). \end{aligned} \tag{44}$$

By (25), for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$ with $\omega^N_i > 0$,

$$t^s_i(c) = c_i(A^s_i(c)) - \frac{1}{\omega^N_i} Z^{\omega^N}(c) + h_i(c_{-i}). \tag{45}$$

Since G^s Pareto-dominates $E^{\gamma^T,\tau}$, for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$ with $\omega_i^N > 0$, by (45), $u(G_i^s(c); c_i) = -\frac{1}{\omega_i^N} Z^{\omega^N}(c) + h_i(c_{-i}) \geq u(E_i^{\gamma^T,\tau}(c); c_i) = -W(c) + \frac{T}{n}$. That is, for each $N \in \mathcal{N}$, each $c \in \mathcal{C}^N$, and each $i \in N$ with $\omega_i^N > 0$,

$$h_i(c_{-i}) \geq \frac{1}{\omega_i^N} Z^{\omega^N}(c) - W(c) + \frac{T}{n}. \tag{46}$$

Let $N = \{i, j\} \in \mathcal{N}$ be such that $\omega_i^N > 2\omega_j^N > 0$. Let $\varepsilon > 0$, $A^1 = (A_i^1, A_j^1) = (\mathbb{A}, \emptyset)$, and $A^2 = (A_i^2, A_j^2) = (\emptyset, \mathbb{A})$. Consider $c \in \mathcal{C}_{ad}^N$ such that

(i) for each $\alpha \in \mathbb{A}$, $c_j(\{\alpha\}) = 2c_i(\{\alpha\}) + \varepsilon$, and

(ii) for each $A = (A_i, A_j) \in \mathcal{A}(N)$ where $A \neq A^2$,

$$\sum_{\alpha \in A_i} c_i(\{\alpha\}) > \frac{1}{\omega_i^N - 2\omega_j^N} \left[F(A^2) - F(A) + \omega_j^N \varepsilon |A_i| \right]. \tag{47}$$

By (45) and (46), for each $l \in N$, $t_l^s(c) \geq c_l(A_l^s(c)) - W(c) + \frac{T}{n}$. Then,

$$\sum_{l \in N} t_l^s(c) \geq \sum_{l \in N} c_l(A_l^s(c)) - nW(c) + T. \tag{48}$$

By $T-$*bounded-deficit*, $T \geq \sum_{l \in N} t_l^s(c)$. This inequality and (48) together imply that

$$nW(c) \geq \sum_{l \in N} c_l(A_l^s(c)). \tag{49}$$

The next Claims follow from (i) and (ii):

Claim 1: $W(c) = \sum_{\alpha \in \mathbb{A}} c_i(\{\alpha\})$.

Proof of Claim 1: Since by (i), for each $\alpha \in \mathbb{A}$, $c_j(\{\alpha\}) > c_i(\{\alpha\})$ and cost functions are *additive*, then $(A_i^\tau(c), A_j^\tau(c)) = A^1$. Hence, $W(c) = \sum_{l \in N} c_l(A_l^1) = c_i(\mathbb{A})$. $\Diamond$

Claim 2: $(A_i^s(c), A_j^s(c)) = A^2$ and $\sum_{l \in N} c_l(A_l^s(c)) = 2 \sum_{\alpha \in \mathbb{A}} c_i(\{\alpha\}) + \varepsilon |\mathbb{A}|$.

Proof of Claim 2: Assume, by contradiction, that $(A_i^s(c), A_j^s(c)) \neq A^2$. Since $c_j \in \mathcal{C}_{ad}$, then $c_j(A_j^2) = c_j(\mathbb{A}) = \sum_{\alpha \in \mathbb{A}} c_j(\{\alpha\})$. By (i),

$$\begin{aligned}
\sum_{l \in N} \omega_l^N c_l(A_l^2) &= \omega_j^N \sum_{\alpha \in \mathbb{A}} c_j(\{\alpha\}) \\
&= 2\omega_j^N \sum_{\alpha \in \mathbb{A}} c_i(\{\alpha\}) + \omega_j^N \varepsilon |\mathbb{A}| \\
&= 2\omega_j^N \sum_{\alpha \in A_i^s(c)} c_i(\{\alpha\}) + 2\omega_j^N \sum_{\alpha \in A_j^s(c)} c_i(\{\alpha\}) + \omega_j^N \varepsilon |\mathbb{A}|. \qquad (50)
\end{aligned}$$

Note also that since $c \in \mathcal{C}_{ad}^N$,

$$\begin{aligned}
\sum_{l \in N} \omega_l^N c_l(A_l^s(c)) &= \omega_i^N c_i(A_i^s(c)) + \omega_j^N c_j(A_j^s(c)) \\
&= \omega_i^N \sum_{\alpha \in A_i^s(c)} c_i(\{\alpha\}) + \omega_j^N \sum_{\alpha \in A_j^s(c)} c_j(\{\alpha\}) \\
&= \omega_i^N \sum_{\alpha \in A_i^s(c)} c_i(\{\alpha\}) + 2\omega_j^N \sum_{\alpha \in A_j^s(c)} c_i(\{\alpha\}) + \omega_j^N \varepsilon |A_j^s(c)|. \qquad (51)
\end{aligned}$$

By (44), $\sum_{l\in N}\omega_l^N c_l(A_l^s(c))+F(A^s(c)) \leq \sum_{l\in N}\omega_l^N c_l(A_l^2)+F(A^2)$. This inequality, (50), and (51) together imply

$$\begin{aligned} \omega_i^N \sum_{\alpha\in A_i^s(c)} c_i(\{\alpha\}) + \omega_j^N \varepsilon |A_j^s(c)| + F(A^s(c)) &\leq 2\omega_j^N \sum_{\alpha\in A_i^s(c)} c_i(\{\alpha\}) + \omega_j^N \varepsilon |\mathbb{A}| + F(A^2) \\ (\omega_i^N - 2\omega_j^N) \sum_{\alpha\in A_i^s(c)} c_i(\{\alpha\}) &\leq F(A^2) - F(A^s(c)) + \omega_j^N \varepsilon (|\mathbb{A}| - |A_j^s(c)|) \\ (\omega_i^N - 2\omega_j^N) \sum_{\alpha\in A_i^s(c)} c_i(\{\alpha\}) &\leq F(A^2) - F(A^s(c)) + \omega_j^N \varepsilon |A_i^s(c)| \end{aligned}$$

which contradicts (47). Hence, Claim 2 must be true. ♢

Now, by Claim 1, $nW(c) = 2\sum_{\alpha\in\mathbb{A}} c_i(\{\alpha\})$. By Claim 2, $\sum_{l\in N} c_l(A_l^s(c)) = 2\sum_{\alpha\in\mathbb{A}} c_i(\{\alpha\}) + \varepsilon|\mathbb{A}|$. Since $\varepsilon > 0$, then $nW(c) < \sum_{l\in N} c_l(A_l^s(c))$ which contradicts (49). This completes the proof for part (b).
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