Critically Asssessing Estimated DSGE Models: A Case Study of a Multi-Sector Model*

X. Liu[†] A.R. Pagan[‡] and T. Robinson[§]

April 30, 2018

Abstract

We describe methods for assessing estimated Dynamic Stochastic General Equilibrium (DSGE) models. One involves the computation of alternative impulse responses from models constrained to have an identical likelihood and the same contemporaneous signs as responses in the DSGE model. Others ask how well the model matches the data generating process; whether there is weak identification; the consequences of including measurement error with growth rates of non-stationary variables; and whether the model can reproduce features of the data that involve combinations of moments. The methods are applied to a large-scale small-open economy DSGE model, typical of those used at policy institutions.

1 Introduction

Estimated Dynamic Stochastic General Equilibrium (DSGE) models today are commonplace both in academia and in policy institutions, such as central banks. An important feature of these models is the definition of some shocks identified from a structural perspective. There are then standard ways in which these models are used. These include the production of impulse response functions for examining policy scenarios and also a decomposition of the observed variables used in estimation into the contributions from each of the shocks. Arguably, there is often much less attention paid to assessing the output of these DSGE models post-estimation, particularly from the perspective of using them for policy analysis. This paper presents a collection of methods that can be used to do so.

*Research Supported by ARC Grant DP160102654. Our thanks to Dan Rees for providing his code.

[†]Department of Economics, University of Melbourne

[‡]School of Economics, University of Sydney and the Centre for Applied Macroeconomic Analysis, Australian National University

[§]Melbourne Institute for Applied Economic and Social Research, University of Melbourne and the Centre for Applied Macroeconomic Analysis, Australian National University. Contact author. E-mail: tim.robinson@unimelb.edu.au.

Throughout the paper we focus our analysis on a multi-sector model of the Australian economy (MSM in this paper) which has been developed by Rees et al. (2016) at the Reserve Bank of Australia. Many of the issues we will address are common to a large number of DSGE models e.g. Andrle (2014) observes that many of the models he has seen feature one of the problems we will observe, namely that empirical shocks are not uncorrelated, even though this was assumed in estimation. Apart from its Australian context, what made the MSM an attractive vehicle for analysis is that it is a fairly large model, and that poses questions that don't arise to the same degree for many of the smaller New Keynesian models one sees in use in academia, largely because they feature only one sector. Moreover, the data and the code used to estimate and simulate it were available. Fundamentally, though we use the MSM to illustrate our general points about DSGE models.

The MSM features seventeen shocks and three production sectors: (i) non-traded commodities and services, (ii) traded non-resource commodities and services, and, (iii) traded resources. It has a unit root process for the log of technology so that some of the variables in the model follow integrated processes that are also co-integrated. What makes the model innovative is the presence of three sectors, and in many ways it can be seen as an extension of the dependent economy model whose origins are strongly Australian - see Metaxas and Weber (2016). A key element in it is a real exchange rate, but it also allows for nominal rigidities, so it is potentially a very useful model for policy analysis. Data on seventeen variables were used to estimate the parameters of MSM. In addition to those commonly used for small-open economy DSGE models, such as GDP growth, inflation, the real exchange rate, and the policy rate, there are a variety of others reflecting sector-specific variables. The foreign sector was captured through three core variables - GDP growth, inflation and a policy rate.[1] An extra variable in MSM that is external to the Australian economy is a resources price. The foreign variables (and resource prices) are strictly exogenous to the Australian economy and so the model is of a small dependent economy. Estimation was performed with Bayesian methods, requiring some prior distributions for the DSGE model parameters to be stated. After estimation some experiments were done with the model in order to assess features such as the impact of monetary and risk premium shocks.

There are many issues raised when assessing output from any DSGE model such as MSM.[2] Section 2 looks at one of these. It stems from the fact that

[1] Throughout this paper we try to use the same notation as in the MSM, with $*$ denoting the foreign sector; for example p and p^* would be the logs of the domestic and foreign price levels respectively. The complete list of variables whose data are used in estimation includes the growth rates in GDP (Δy^{va}), consumption (Δc), investment (Δi), public demand (Δg), resource exports (Δz^x), non-resource exports (Δy^x_m), non-tradeable value added Δy^{va}_n, non-resource tradeable value added (Δy^{va}_m), resources valued added (Δy^{va}_z), domestic inflation π_t, non-tradeable inflation (π_n), the Australian cash rate (r) and the change in the nominal exchange rate (Δs). There are also data on growth in foreign GDP (Δy^*), inflation (π^*), a short term interest rate (r^*) and resource price inflation (Δp^*_z).

[2] The approaches adopted here are not meant to be exhaustive - see, for example, Schorfheide (2013) for an earlier discussion. The methods we advance can be thought of

shocks are a pivotal feature of DSGE models and that one can recombine them to produce different *different impulse responses* and *yet have the same fit* as the estimated DSGE model.[3] When using the latter for policy scenarios we would presumably want to know how big the range of alternative impulse responses is. We find such alternative models, after imposing two restrictions. First, the alternative model must fit the data equally well as the DSGE model. Second, the impulse responses to the named shocks from any alternative model must have the same contemporaneous signs as those given by the estimated DSGE model, in this case the MSM.[4] To perform this task we utilize the fact that a DSGE model solves for a Vector Autoregression in all its variables, and this can be written in such a way as to highlight its structural shocks. We refer to the resulting representation as a *semi-structural VAR (SSVAR)* model. Because many DSGE models, including MSM, imply that there is co-integration between certain variables, Section 3 also considers different representations of the basic SSVAR from MSM, moving towards a semi-structural Vector Error Correction. Such a representation is useful for a number of analyses of output from DSGE models such as MSM.

Now a model can be defined as a representation of a system that allows an investigation of the properties of that system. A model is built up from what are thought to be key variables in the system, and the quantitative relationships between then are captured by assigning some values to parameters involved in these. Hence a different set of parameter values implies a different model, although they may not imply very different properties for the system. Think of a demand/supply system. The model and implied system properties are different if there is a very low supply elasticity to when there is a high one, and the model differences will be evident from a graph. Hence, multiple models with the same generating process can arise if there are many values for the estimated parameters which produce the same likelihood. If so, then there would be many different values for impulse responses. This would be an example of a failure of structural identification of the DSGE model parameters, and so it is important to check for such problems. Basically, one wants to check the shape of the likelihood (or whatever function is being optimized to produce parameter estimates). Some identification measures are now in Dynare and these were set out in Ratto and Iskrev (2010). In Section 4 we discuss these measures and apply them to the MSM model. Another approach,described in

as complements to existing techniques. A lot of the previous literature focusses on judging either whether the DSGE model fits the data as well as alternatives such as VARs or whether it will produce good predictions. Although we do have some suggestions for judging the quality of fit they are based on examining measures that have some particular interest e.g. on whether recessions are more likely to be predicted by the model rather than are present in the data.

[3] Although we focus on DSGE models what we propose can also be used for many other structural macroeconomic models. What is distinctive about DSGE models however is their concentration upon shocks.

[4] Of course if one insisted that the impulse responses must have the same *quantitative* values as those from the DSGE model then there should be no alternative models if the model is identified.

Koop et al. (2013), uses a convergence rate indicator to flag parameters that are possibly only weakly identified. This involves simulation of the estimated DSGE model and, in Section 4, we suggest another way that the simulated data can be fruitfully used to judge identification issues. It appears that some of the crucial parameters in the MSM, namely the slopes of Phillips curves, may be weakly identified.

Finally, we have the fundamental question of how the generating process of the selected DSGE model (the *model generating process MGP*) matches the *data generating process (DGP).* That there can be a gap between these comes from the fact that some of the DSGE model variables may be unobserved i.e. do not have an exact analogue in the data. In that situation, although the MGP for all variables may be a VAR, this may not be true of the DGP for the observed variables. Section 5 looks at this in a number of stages. Given a distinction between the data and the model generated variables, it seems natural to ask how one might bridge these? In many DSGE models, including MSM, reconciling the model and data is often partially done by allowing for "measurement error" in the data. Section 5.1 examines how productive this approach is, pointing out the difficulties with it when data is measured by growth rates in I(1) variables, as is done in MSM.

Section 5.2 moves on to ask whether dropping the unobservables means that the form of the MGP for the complete MSM differs from the format of the generating process for a reduced number of variables. Using impulse responses to measure the correspondence between these two generating mechanisms we find that the SSVAR(2) implied by the MSM is reasonably well approximated by an SSVAR(2) in just observables. In Section 5.3 we ask how well certain features of the MGP match the DGP of the observables. This is a quantitative test. Simple statistics such as moments can be informative about this question, and more complex ones, such as business cycle outcomes, can be important for conceptualizing what any failure to match the data means. Finally, at various times we utilize the different representations of Section 3 in order to shed light on a failure of data and the MSM model moments to match.

2 Examining the Model Generating Process of MSM

2.1 Generating a Range of Models Compatible with a Given Model Generating Process

Let the DSGE model have parameters θ. Then its variables will be generated using the model with these parameters set to some estimated values θ^*. We will call this the *Model Generating Process (MGP).* Suppose there are other models that have the same generating process as that found with θ^*, i.e. the MGP, but with different impulse responses to shocks. In that case, the location of the responses of the DSGE model in this range is a useful indicator of the uncertainty surrounding such responses when used for policy analysis. This is *different to the statistical uncertainty* coming from the fact that the parameters are estimated. Rather it is *model uncertainty*, reflecting the fact that there are other models compatible with the MGP but which produce different reactions to shocks.[5] One way to think about the difference is to suppose we have an infinite amount of data. Then the statistical uncertainty would disappear but the model uncertainty would remain. The focus of this sub-section is to demonstrate how it is possible to quantify the extent of the latter.[6] In particular, we show how to produce such a range of models and define what we mean by compatibility.

Variables in DSGE models can be taken to be $I(0)$, perhaps after some transformation. The most common transformation needed is to convert $I(1)$ variables into $I(0)$ variables by de-trending them with the level of technology. We will focus on this later but, for now, assume that the z_t in a DSGE model are all $I(0)$ variables. In most instances the DSGE model has the structural equations[7]

$$A_0 z_t = C E_t(z_{t+1}) + A_1 z_{t-1} + H u_t, \tag{1}$$

where u_t are shocks possibly following a VAR(1), $u_t = \Phi u_{t-1} + \varepsilon_t$, and ε_t is a vector of white noise structural shocks with covariance matrix Σ that is diagonal. The latter are generally referred to as innovations to the structural shocks and we will use that terminology here. A_0, C, A_1 and H are matrices which are functions of θ. This system can then be solved for z_t by using (for example) the method of undetermined coefficients, and it produces a solution

$$z_t = B z_{t-1} + G u_t.$$

Binder and Pesaran (1995) present the two relevant conditions for this solution to exist, namely a rank condition and the Blanchard-Kahn stability conditions. Hence

$$\begin{aligned} z_t &= B z_{t-1} + G(\Phi u_{t-1} + \varepsilon_t) \\ &= B z_{t-1} + G\Phi(G^+(z_{t-1} - B z_{t-2})) + G\varepsilon_t \\ &= (B + G\Phi G^+) z_{t-1} - G\Phi G^+ B z_{t-2} + G\varepsilon_t \\ &= B_1 z_{t-1} + B_2 z_{t-2} + e_t, \end{aligned}$$

where $e_t \equiv G\varepsilon_t$ are the VAR error terms, G^+ is the (possibly generalised) inverse of G, $B_1 \equiv (B + G\Phi G^+)$ and $B_2 \equiv -G\Phi G^+ B$. This is a VAR(2) in which the VAR errors have been written as functions of the structural shocks ε_t. For convenience we will refer to this as a *semi-structural VAR (SSVAR)*. As we will see later the MSM can be expressed in this form.

[5] It is important to note that we are not discussing alternative models that have a different set of variables in them compared to the DSGE model. A lot of work on specification errors in DSGE models involves expanding the set of variables used.

[6] The presentation of the solution of the DSGE model draws on Pagan and Robinson (2016).

[7] DSGE models with long lags can be accommodated by expanding z_t to include lagged variables. The subsequent analysis is similar; the current values of variables would have to be selected from z_t.

Now let us write

$$\begin{aligned} z_t &= B_1 z_{t-1} + B_2 z_{t-2} + G\Sigma\Sigma^{-1}\varepsilon_t \\ &= B_1 z_{t-1} + B_2 z_{t-2} + G\Sigma\eta_t, \\ &= B_1 z_{t-1} + B_2 z_{t-2} + F\eta_t, \end{aligned}$$

where $F \equiv G\Sigma$. The resulting η_t will have unit variances but the impulse responses to η_t *are the same as those for a one standard deviation perturbation to whatever the shocks* ε_t are named.

That there are other models with different structural impulse responses can be seen by writing

$$\begin{aligned} z_t &= B_1 z_{t-1} + B_2 z_{t-2} + FQ'Q\eta_t \\ &= B_1 z_{t-1} + B_2 z_{t-2} + D\tilde{\eta}_t, \end{aligned}$$

where Q is a matrix with the property that $QQ' = Q'Q = I, D = FQ'$ and $\tilde{\eta}_t = Q\eta_t$.[8] Then the new shocks $\tilde{\eta}_t$ will be uncorrelated and the impulse responses to them will also be to a one standard deviation perturbation in whatever they are named. So the contemporaneous impulse responses have been changed from F to D, i.e. we have a new model.

In what sense is the new model compatible with the MGP of the original model with shocks η_t? The answer is that, since the $cov(e_t) = cov(G\Sigma\eta_t) = cov(G\Sigma Q'\tilde{\eta}_t)$, the model with $\tilde{\eta}_t$ shocks produces the same covariance matrix for the VAR errors. Because B_1 and B_2 have not changed, the density function for z_t must be the same for both models i.e. the likelihood has not changed. The new and existing model fit the data equally as well.[9]

Alternative Q matrices will therefore be the approach used to study the range of impulse responses that are compatible with the MSM model, which is a way to quantify the extent of model uncertainty present. The nature of this analysis with the semi-structural VAR has strong parallels with sign-restricted VARs, but B_1 and B_2 here are anchored by the DSGE model.

2.2 The Nature of Shocks in the Estimated MSM Model

Before we proceed to further analysis it is necessary to make clear what the constraints are when generating any new set of impulse responses. Specifically, we *will not be generating* impulse responses that produce a *better match* to the data. Instead, we will be generating responses that constrain the shocks to be uncorrelated and which replicate certain hypothetical results from the MSM model. In these hypothetical results the dynamic parameters of the model B_1

[8] Fry and Pagan (2011) discuss Q matrices that have this property. The best known of them is the Givens matrix used in the sign restriction literature by Canova and de Nicolo (2002) while Rubio-Ramirez et al (2010) give a general way of finding a Q matrix with the requisite properties using simulation methods. We use an adaption of that method in what follows.

[9] In this respect the SSVAR is distinctly different to the DSGE-VAR literature, such as Del Negro and Schorfheide (2004).

and B_2 are fixed at values estimated from data, while the structural shocks are *assumed* to be uncorrelated. We raise this issue since it is not the case that the MSM shocks estimated from the data are uncorrelated. Table 1 shows some of the larger correlations.[10]

Table 1 Correlations of Selected Shocks from the Estimated MSM Model

Shock Pair	Correlation
corr$(\varepsilon_r, \varepsilon_{y^*})$	.67
corr$(\varepsilon_{r^*}, \varepsilon_{\psi})$	-.36
corr$(\varepsilon_{\pi^*}, \varepsilon_{\psi})$	.44
corr$(\varepsilon_{p^*}, \varepsilon_r)$	.34
corr$(\varepsilon_g, \varepsilon_{\Upsilon})$	-.56
corr(ε_{r^*}, μ)	-.53
corr$(\varepsilon_f, \varepsilon_n)$	.78
corr$(\varepsilon_{m^*}, \varepsilon_{a_m})$	.62
corr$(\varepsilon_m, \varepsilon_n)$	.32

To understand why Table 1 can record such non-zero correlations, even though they are assumed zero in estimation, take the two regression equations

$$\begin{aligned} y_{1t} &= x_{1t}\beta_1 + \varepsilon_{1t} \\ y_{2t} &= x_{2t}\beta_2 + \varepsilon_{2t}, \end{aligned}$$

and *assume* that ε_{1t} and ε_{2t} are uncorrelated when estimating β_1 and β_2. Then, based on that assumption, the maximum likelihood estimates of β_1 and β_2 would be the OLS estimates. However, nothing guarantees that the residuals $\hat{\varepsilon}_{jt}$ formed from these are orthogonal. For this to be asymptotically true it would be necessary that the assumption of zero correlation between the shocks is correct. If this assumption is not true in the data, then $\hat{\varepsilon}_{jt}$ will not be orthogonal. Of course the estimators of β_j in the example above are consistent, even if there is correlation.

The reason for the lack of orthogonality is that we have two moment conditions defining the parameter estimates β_j- $E(x_{jt}\varepsilon_{jt}) = 0, (j = 1, 2)$ - and two others defining the shock variances - $E(\varepsilon_{jt}^2) = \sigma_j^2$ $(j = 1, 2)$ - meaning that the moment condition $E(\varepsilon_{1t}\varepsilon_{2t}) = 0$ has not been needed for estimation of the parameters. So the moment conditions deliver more than is needed to get estimators of the parameters i.e. we have an over-identified model.

Suppose instead we had written

$$y_{1t} = x_{1t}\beta_1 + \varepsilon_{1t} \tag{2}$$

$$y_{2t} = x_{2t}\beta_2 + \gamma y_{1t} + \varepsilon_{2t}. \tag{3}$$

Then three moment conditions are needed to estimate all the unknown parameters, the system is *exactly identified*, and the residuals are orthogonal.

[10] The symbols are as in the MSM.

The last scenario is a characteristic of recursively just-identified SVAR models, and it is this that ensures shocks in such models are orthogonal. In contrast, DSGE models are typically heavily *over-identified*. For example, in Smets and Wouters (2007), 36 parameters are estimated with 7 observed variables; a just-identified SVAR(2), in contrast, would include 126 estimated parameters. In MSM it is even more stark; 45 parameters are estimated, but there are 600 moment conditions. The theory built into these models results in them often being very tightly parameterised.

In an overidentified system, not all of the moment conditions are used in estimation or can be satisfied at once. Consequently, even though it is *assumed that the shocks are uncorrelated*, this restriction may not be used in estimation and the shocks obtained from an estimated DSGE model may well be correlated.[11] This provides one explanation of the results in Table 1.

A second, supplementary, reason involves the use of Bayesian estimation. Take the model in (2) and (3) but set $x_{1t} = x_{2t} = z_t$. Let the MLE(OLS) estimates by $\hat{\beta}_j$,$\hat{\gamma}, \hat{\sigma}_j$ and the Bayesian estimates be β_j^B, γ^B. Then we know that the OLS residuals $\hat{\varepsilon}_{1t}$ and $\hat{\varepsilon}_{2t}$ satisfy the condition $\frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_{1t}\hat{\varepsilon}_{2t} = 0$, where $\hat{\varepsilon}_{1t} = y_{1t} - z_t\hat{\beta}_1$, $\hat{\varepsilon}_{2t} = y_{2t} - \hat{\gamma}y_{1t} - z_t\hat{\beta}_2$. This is a zero covariance between estimated shocks.[12] Writing $\hat{\beta}_j = \beta^B + (\hat{\beta}_j - \beta^B), \hat{\gamma} = \gamma^B + (\hat{\gamma} - \gamma^B)$ we can see that the sample covariance between the Bayesian shocks is

$$\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{1t}^B\varepsilon_{2t}^B = \hat{\sigma}_1^2(\hat{\gamma} - \gamma^B) + (\hat{\beta}_1 - \beta_1^B)\hat{\sigma}_z^2[\hat{\beta}_1(\hat{\gamma} - \gamma^B) + (\hat{\beta}_2 - \beta_2^B)],$$

where $\varepsilon_{1t}^B, \varepsilon_{2t}^B$ are the estimated Bayesian shocks using β_j^B, γ^B instead of $\hat{\beta}_j, \hat{\gamma}$. There is no reason why this should be zero unless $\hat{\beta}_j = \beta_j^B, \hat{\gamma} = \gamma^B$.[13] Now the Bayesian estimates of parameters, β_j^B, γ^B can be thought of as a weighted average of the MLEs and the priors for β_j, γ. So it is unlikely that $\hat{\beta}_j = \beta_j^B$, $\hat{\gamma} = \gamma^B$ although if the sample size is large the prior will drop out and these will converge to zero. So another possible explanation is that Table 1 reflects Bayesian estimation.[14] Nevertheless, because the MSM is highly over-identified it seems more likely that the results are due to that feature.

[11] Andrle, M. (2014) has made this point as well and says - "Yet, the actually estimated 'structural' shocks are strongly correlated as a rule rather than exception. Correlated structural shocks are a sign of misspecification". As our simple example shows this need not be the case, although it is certainly inconsistent with the statistical assumptions being made for estimation.

[12] We assume here that all data are mean corrected - as is common in DSGE models - so OLS residuals have zero mean.

[13] The point we are making here is that in exactly identified models MLE will produce orthogonal shocks but the use of prior information will change this. It is not an issue of having a small sample. The OLS shocks have a zero correlation for any sample size but the Bayesian ones will only have this for large samples.

[14] We have looked at the shocks in the Smets and Wouters (2007) model and find that there the shocks are correlated but the correlations are smaller than for the MSM, probably because the degree of over-identification is very much smaller. Of the 21 shock correlations 4 exceed .2 in absolute value.

It should be said that this is not the only problem that can arise for model shocks from over-identifying information. The restriction that the *innovations* to the structural shocks have no serial correlation may not be enforced. Table 2 shows that this is clearly evident for the MSM; for example the supposed innovation to technology growth actually has an autocorrelation coefficient of 0.64. The phenomenon is not isolated to the innovations for the structural shocks, but can also be true for any of the "measurement errors" that we will discuss later. As Table 2 shows the autocorrelation of that type of shock for resource prices is particularly high (0.76). So the empirical shocks and the measurement errors are often very different from what has been assumed about them. When it comes to data, assuming that they are white noise does not make them so, as regression users learnt many years ago.

Table 2 Autocorrelation of Selected Shock Innovations from the Estimated MSM Model

Innovation	**Mnemomic**	**Autocorrelation**
Technology Growth	ε_{μ}	.64
Investment Efficiency	ε_{Υ}	-.33
Resource Price	$\varepsilon_{p_z^*}$	.40
Foreign Output	ε_{y^*}	.64
Monetary Policy	ε_{r}	.59
Measurement Error		
Nominal Exchange Rate		.39
Foreign GDP		.52
Resource Price		.76

What are the consequences of the shocks being correlated? From the perspective of using the model, it makes variance decompositions problematic. Moreover, one also really needs shocks to be innovations in order to utilize impulse responses, for these work under a *ceteris paribus* assumption that would not hold when shocks are correlated. Thus, if two shocks, say money and technology, are correlated, then we don't know what the common component to them represents - is it money or technology? Decompositions of variables according to shocks are then very hard to interpret.

Of course we can take the estimated parameters B_1, B_2 as given and then ask what the impulse responses are in this hypothetical context, that is *assuming* that the shocks are uncorrelated, even though this assumption may not be compatible with the data. The variance decompositions presented in Rees et al. (2016) are a hypothetical experiment and relate to internal consistency of the model, as the experiment does not represent the situation with the estimated shocks and the data variables. Essentially they are constructing a scenario i.e. asking what the results would be if the estimated parameter values were used for B_1 and B_2, and then uncorrelated shocks are applied, rather than the actual shocks found from the data. In what follows we will adopt this same scenario, constructing models and impulse responses which utilize the estimated B_1, B_2 and which reproduce the $cov(e_t)$ from the experiments they performed.

2.3 Constructing the Alternative Models in Practice

One difficulty in constructing alternative models that have the same fit as the hypothetical MSM model, but which embody different impulse responses, is to make decisions about which shocks should be recombined. To understand the issue consider the three variable SSVAR given by (4)- (6) that we wish to capture with an alternative model:

$$\begin{aligned} y_{1t} &= b_{11}^1 y_{1t-1} + b_{12}^1 y_{2t-1} + b_{13}^1 y_{3t-1} + f_{11}\eta_{1t} && (4) \\ y_{2t} &= b_{21}^1 y_{1t-1} + b_{22}^1 y_{2t-1} + b_{23}^1 y_{3t-1} + f_{21}\eta_{1t} + f_{22}\eta_{2t} + f_{23}\eta_{3t} && (5) \\ y_{3t} &= b_{31}^1 y_{1t-1} + b_{32}^1 y_{2t-1} + b_{33}^1 y_{3t-1} + f_{31}\eta_{1t} + f_{32}\eta_{2t} + f_{33}\eta_{3t}. && (6) \end{aligned}$$

In this model only one shock affects y_{1t}. Now consider constructing new shocks $\tilde{\eta}_{jt} = q_{j1}\eta_{1t} + q_{j2}\eta_{2t} + q_{j3}\eta_{3t}$ using a Q matrix. Then we see that all of the new shocks $\tilde{\eta}_{jt}$ will have an impact upon y_{1t}, because they are formed with η_{1t}. However, we may not want all of the $\tilde{\eta}_{jt}$ shocks to impact upon y_{1t}, as this would not preserve the fact that only one of the $\tilde{\eta}_{jt}$ should have a non-zero impact on it. Therefore a Q matrix must be designed that preserves the zero impact of some shocks. The simplest way to do this is to set $q_{11} = 1, q_{12} = 0, q_{13} = 0, q_{21} = 0$ and $q_{31} = 0$. Consequently, $\tilde{\eta}_{1t} = \eta_{1t}$ and $\tilde{\eta}_{jt} = q_{j2}\eta_{2t} + q_{j3}\eta_{3t}$ $(j = 2, 3)$. This means that we use a (3×3) Q matrix of the form

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & Q_2 \end{bmatrix},$$

where the (2×2) matrix Q_2 has the properties $Q_2'Q_2 = I_2 = Q_2Q_2'$. Clearly Q has the requisite properties of $Q'Q = I_3 = QQ'$.[15]

Another issue arises with the external sector. The MSM has a simple New-Keynesian representation for the foreign sector of the form

$$\begin{aligned} \tilde{y}_t^* &= E_t(\tilde{y}_{t+1}^*) - (r_t^* - E_t(\pi_{t+1}^*)) + u_{yt}^* && (7) \\ \pi_t^* &= \beta E_t(\pi_{t+1}^*) + \kappa \tilde{y}_t^* + u_{\pi t}^* && (8) \\ r_t^* &= \rho_r r_{t-1}^* + (1-\rho_r)(\gamma_y \tilde{y}_t^* + \gamma_\pi \pi_t^*) + \delta \Delta \tilde{y}_t^* + \varepsilon_{rt}^*. && (9) \end{aligned}$$

Here the variables in the model are foreign ones and so are distinguished with an asterisk. u_{yt}^* and $u_{\pi t}^*$ are AR(1) shocks driven by innovations ε_{yt}^* and $\varepsilon_{\pi t}^*$, y_t^* is the log level of foreign output, $\tilde{y}_t^* = y_t^* - a_t$, where a_t is the log of the level of technology, and π_t^*, r_t^* are the foreign inflation and interest rate. There is a separate equation, namely

$$\Delta y_t^* = \Delta \tilde{y}_t^* + \Delta a_t = \Delta \tilde{y}_t^* + \mu_t, \qquad (10)$$

[15] Arias et. al. (2014) have another way of generating Q that preserves zero restrictions. Consequently, our choice may not exhaust the range of possible Q matrices and alternative models that might be generated, but that would only mean that our range is possibly smaller. Because it turns out that our range is large this does not seem a crucial issue, but further investigation is warranted.

where μ_t is the innovation into technology described in the MSM model. We will not re-combine this technology shock μ_t with the other external shocks in forming new shocks. The reason is that μ_t is a permanent shock and the other three are transitory. As Fry and Pagan (2011) observed one cannot combine permanent with transitory shocks if you want some of the final shocks $\tilde{\eta}_t$ to be transitory. Given that all shocks (apart from technology) are transitory in MSM, it is not sensible to include the technology shock in the set to be re-combined.[16]

Finally, we have foreign and domestic shocks. In the MSM Australia is assumed to be a small open economy, that is, the foreign sector is strictly exogenous. However, if some of the newly created shocks were obtained by combining the MSM external and domestic shocks then the small open economy assumption would be violated. Consequently, by constructing new uncorrelated shocks from the hypothetical uncorrelated shocks of the MSM those shocks that are not re-combined are left at the MSM estimates, and so they will be uncorrelated with any combination of the other shocks.

The MSM external sector has a SSVAR(1) format

$$z_t = B_1 z_{t-1} + \varepsilon_t^{MSM},$$

where $z_t' = \left[\begin{array}{ccc} \tilde{y}_t^* & \pi_t^* & r_t^* \end{array} \right]$. Hence, using the parameter values given by Rees et al. the solution to the external system is an SSVAR(1) of the form[17]

$$\tilde{y}_t^* = .865\tilde{y}_{t-1}^* - .248\pi_{t-1}^* + .083r_{t-1}^* + .003\varepsilon_{yt}^{*MSM} - .005\varepsilon_{\pi t}^{*MSM} - .049\varepsilon_{rt}^{*MSM} \quad (11)$$

$$\pi_t^* = .108\tilde{y}_{t-1}^* + .269\pi_{t-1}^* + .123r_{t-1}^* + .0006\varepsilon_{yt}^{*MSM} + .013\varepsilon_{\pi t}^{*MSM} - .009\varepsilon_{rt}^{*MSM} \quad (12)$$

$$r_t^* = .954r_{t-1}^* + .005\tilde{y}_{t-1}^* - .0095\pi_{t-1}^* + .0005\varepsilon_{yt}^{*MSM} + .0005\varepsilon_{\pi t}^{*MSM} + .001\varepsilon_{rt}^{*MSM}. \quad (13)$$

The equations (11)-(13) are identities. The shocks, such as ε_{yt}^{*MSM}, can be converted to the corresponding unit variance shocks, η_{yt}^*, simply by re-scaling the coefficients attached to ε_{yt}^{*MSM} by the standard deviations of ε_{yt}^{*MSM} etc. As an example, the interest rate equation becomes

$$r_t^* = .954r_{t-1}^* - .0095\pi_{t-1}^* + .005\tilde{y}_{t-1}^* + .000098\eta_t^{r^*} + .0001\eta_t^{\pi^*} + .0065\eta_t^{y^*}. \quad (14)$$

Then a new set of shocks $\tilde{\eta}_t^*$ needs to be constructed that are linear combinations of the original normalised shocks, $\eta_t^{y^*}$ etc.. This must be done in such a way as to ensure that they are uncorrelated with unit variance. To be clear, the new SSVAR will still have the same dynamics i.e. B_1 is fixed at the MSM estimated values, and the covariance matrix for the hypothetical reduced-form errors will be replicated by the new set of innovations $\tilde{\eta}_t^*$. Consequently the

[16] If we combined μ_t with the transitory shocks to create new ones $\tilde{\eta}_t$, these would have permanent effects and that would mean there is more than one supply-side shock.

[17] This was obtained using the simulation method in Pagan and Robinson (2016).

alternative model fits the data equally well as the MSM external sector specification. However, these new shocks $\tilde{\eta}_t^*$ will have different impulse responses, and it is useful to look at the range of responses that can be generated. Of course a wider range of impulse responses might be found that produce a superior fit by allowing changes in the dynamics as well. Our focus here, however, is to gauge the extent of model uncertainty for a given fit with the data.

Some of the alternative models that we generate with uncorrelated shocks might be ruled out. This could be because they produce responses of unrealistic magnitude. A weaker constraint is to eliminate models that do not produce the same signs for contemporaneous impulse responses as the MSM does. Table 3 gives the latter for (positive) shocks to the structural equations (7)-(9).

Table 3: Signs for Contemporaneous Impulse Responses					
To Three Shocks in the MSM External Sector					
Variable		**Shocks**			
		Demand	**Cost**	**Interest Rate**	
$\tilde{y}^*$		>0	<0	<0	
π^*		>0	>0	<0	
r^*		>0	>0	>0	

The contemporaneous impulse response functions for the new shocks will be compared to the signs from those in the MSM given in Table 3. If they agree the impulse responses corresponding to the new shocks are accepted. If they don't then we draw a new Q, and once again combine together the three MSM shocks. It is important to emphasize that we have retained the MSM dynamics in this operation, i.e. B_1 is fixed.

Generating 1000 models with uncorrelated shocks by re-combining the MSM ones we find that 118 satisfy the sign restrictions in Table 3. Now, doing this means we will be finding impulse responses in the alternative models to a one standard deviation perturbation. But in each model there will be a different standard deviation for a shock such as the monetary one. Consequently we need to re-scale these impulse responses so that they are comparable to the MSM results (we will refer to these as the *standardized shocks*).[18]

Focussing on the results for a monetary policy shock, these are shown in Table 4. Considering first the contemporaneous impact on output, it is apparent that the MSM responses are very much at the high end of the scale for monetary effects. In fact there are only three impulse responses of the 118 that are larger (in absolute terms) than that given by the estimated MSM. Therefore, it is possible to find models that are observationally equivalent to MSM (in the sense of replicating second moments) but which deliver a much lower impact for monetary policy. This is the type of result that can be produced from the semi-structural VAR approach that we believe will be of interest to policymakers when assessing the output produced from estimated DSGE models. At the

[18] This requires us to find the implied standard deviation of the monetary shock in the alternative models. To do this we use the method given in Ouliaris and Pagan (2016).

moment one is using a model that implies a very strong impact for foreign monetary policy when many other models could be produced that have the same dynamics but a much weaker response.

Table 4: Contemporaneous Responses to a Standardized Monetary Policy Shock in the MSM Foreign Sector

Magnitude	Variables	
	Output	**Inflation**
Maximum	-.437	-.276
Minimum	-.0004	-.0001
MSM	-.346	-.066

Turning to inflation, it appears that the result predicted by MSM is less extreme. As has been argued by Ouliaris and Pagan (2016), *inter alia,* it is useful to think of the average of the maximum and minimum values as a representative value. With that choice the MSM results are found to be about 1/2 of the representative value.[19] These results suggest that one either might need to look more closely at the specification of the New-Keynesian model which is at the heart of the foreign sector of MSM or at least perform experiments with the equivalent models that produced the minimum and maximum responses. Of course based on the magnitudes of the responses one might be able to rule out some of the range of estimates.[20]

Now it is important to emphasize that we have only considered alternative models that have the impulse response signs of Table 3 for a limited range of variables. But the MSM also produces impulse responses of domestic variables to these external shocks and they have a set of signs implied by the MSM model. Hence we might reject an alternative model if it fails to reproduce the contemporaneous signs of the impulse responses of *all variables*, both domestic and foreign. When we do this some of the models above will be rejected. Indeed, of the 118 models found above, only 3 are now retained as agreeing with all the signs for impulse responses. Therefore, this requires the generation of many more models than the 1000 used before in order to study the range of impulse responses that are possible. Accordingly, we generated 100000 models, 77 of which were retained as providing a complete match with the signs of all the impulse responses from the MSM (ignoring the level of government expenditure variable where there are zero effects of all shocks). Now the biggest and smallest effects of monetary shocks on outout are at -.0353 and -.001, so the range has

[19] Baumeister and Hamilton (2015) pointed out that statistics such as the median of the range of outcomes depended upon the simulation method employed and so were not especially informative. This is illustrated in a simple way in Ouliaris and Pagan (2016).

[20] Regarding the foreign interest rate identity, (14) shows that the foreign demand shock is the dominant force in the evolution of the foreign interest rate. Indeed, in the variance decomposition for the external system the foreign demand shock η_{y^*} explains 99.12% of the variance of the foreign interest rate, while the monetary shock explains virtually nothing. This seems a little odd. Furthermore, 41% of the foreign output gap variance is due to the monetary policy shock, an impact of a magnitude rarely seen in small New-Keynesian models.

narrowed, but it is still the case that the MSM output response is virtually the largest in the complete set of alternative models. The same outcome is true of the inflation response, where the minimum is now -.0359 and the maximum is -.1072.[21]

3 The Nature of Variables and Their Representation

3.1 The Nature of Variables

In DSGE models, particularly as they become large, many of their variables have no counterpart in the data used to estimate their parameters. A useful distinction that can be made between the variables is to separate them into the following categories: (i) observable, i.e. data is available on them; (ii) partially observable, i.e. some data is available which contains information about them; (iii) redundant, namely they can be substituted out as a function of other variables, and (iv) strictly unobservable, for which there is no equivalent data.

Turning to MSM, an example of an observed variable is the interest rate r_t, while stationized GDP, $\tilde{y}_t^{va}$, is partially observed through GDP growth. In total, the MSM model contains more than 80 variables, of which 55 are redundant. Consequently, in the SSVAR representation of the MSM discussed above there were 24 variables, of which 17 are observable or partially observable and seven are unobserved. Examples of the latter are the net foreign assets to GDP ratio and the sectoral capital stocks.[22] MSM also includes 17 shocks (excluding measurement errors).

3.2 A VAR/VECM Representation of DSGE Models Via Error-Correction terms

Many DSGE models now include permanent shocks, such as non-stationary technology, and therefore they feature cointegration between many of the variables. This is true of MSM. Our objective here is to look at what the Vector Error Correction representation of it might be. This was done theoretically for DSGE models in Christensen et al.. (2011), but it is much simpler to use the simulation approach in Pagan and Robinson (2016) to construct the underlying SSVAR. Indeed what we will develop could be called a semi-structural VECM, since the system will include changes in some variables, as well as error-correction terms. There are other aspects to reconciling the model and the data,

[21] In Liu et al. (2018) we also look at the range of the MSM impulse responses to domestic shocks that might be generated and retain the same fit as the MSM, finding that that they seem to be less extreme. The method of doing this is the same as for the foreign sector so is omitted to conserve space.

[22] The complete list of unobserved variables includes $\pi_{f,t}, b_t^*, k_{m,t}, k_{z,t}, \lambda_{z,t}, \lambda_{n,t}$ and $\lambda_{m,t}$.

such as the nature of deterministic trends and their implications for the error-correction terms. In the MSM model the deterministic growth rates in all of the real variables are the same. However, this may not be true in the data. To handle this problem Rees et al. adopted a commonly used approach, namely they mean corrected the data growth rates and then modelled the resulting series. Essentially this removes a linear deterministic trend from the log levels of data and thereby avoids any lack of co-trending between the variables. It means that the error-correction terms should not have any trend. However, if there are breaking trends in the data, rather than a constant one, this can show up as a trend in the error-correction terms. Such a breaking trend would need to be allowed for in estimation, otherwise there is a mis-specification and the likelihood is incorrect.

As many of the policy-oriented DSGE models such as MSM are large, in order to demonstrate the semi-structural VECM representation we first consider a simple example. Suppose we had a DSGE model that had three variables that were $I(1)$ - the logs of domestic output y_t, consumption c_t, and foreign output y_t^*. Then these are integrated processes because of the log level of technology a_t being $I(1)$. Just as for the MSM, in this DSGE model we would have variables $\tilde{y}_t = y_t - a_t, \tilde{c}_t = c_t - a_t$ and $\tilde{y}_t^* = y_t^* - a_t$. Imposing strong exogeneity of the foreign sector the SSVAR would have a form such as

$$\begin{aligned}
\tilde{y}_t^* &= b_{11}\tilde{y}_{t-1}^* + \varepsilon_{y_t^*} \\
\tilde{y}_t &= b_{21}\tilde{y}_{t-1}^* + b_{22}\tilde{y}_{t-1} + b_{23}\tilde{c}_{t-1} + g_{21}\varepsilon_{y_t^*} + g_{22}\varepsilon_{yt} + g_{23}\varepsilon_{ct} \\
\tilde{c}_t &= b_{31}\tilde{y}_{t-1}^* + b_{32}\tilde{y}_{t-1} + b_{33}\tilde{c}_{t-1} + g_{31}\varepsilon_{y_t^*} + g_{32}\varepsilon_{yt} + g_{33}\varepsilon_{ct}.
\end{aligned}$$

Now there are four $I(1)$ variables here - y_t, c_t, y_t^* and a_t - and there are three error-correction terms - $\tilde{y}_t, \tilde{c}_t$ and $\tilde{y}_t^*$. Rather than use this SSVAR form we want to rewrite the equations above in terms of observable error-correction terms, since $\tilde{y}_t$ and $\tilde{c}_t$ are only partially observable due to the technology shock. It should be noted, however, that it is not possible to write the model in terms of *observable* error-correction (EC) terms alone.[23] We will use two observable ones, namely $\xi_{1t} = y_t - y_t^*$ and $\xi_{2t} = c_t - y_t$, and one partially observable, $\xi_{3t} = \tilde{y}_t^*$. The equation for $\Delta\tilde{y}_t$ can then be expressed as

$$\begin{aligned}
\Delta\tilde{y}_t &= b_{21}\tilde{y}_{t-1}^* + (b_{22}-1)\tilde{y}_{t-1} + b_{23}\tilde{c}_{t-1} + g_{21}\varepsilon_{y_t^*} + g_{22}\varepsilon_{yt} + g_{23}\varepsilon_{ct} \\
&= b_{21}\tilde{y}_{t-1}^* + (b_{22}+b_{23}-1)\tilde{y}_{t-1} + b_{23}\xi_{2t-1} + g_{21}\varepsilon_{y_t^*} + g_{22}\varepsilon_{yt} + g_{23}\varepsilon_{ct} \\
&= (b_{21}+b_{22}+b_{23}-1)\tilde{y}_{t-1}^* + (b_{22}+b_{23}-1)\xi_{1t-1} + b_{23}\xi_{2t-1} + g_{21}\varepsilon_{y_t^*} \\
&\quad + g_{22}\varepsilon_{yt} + g_{23}\varepsilon_{ct} \\
&= \phi_{21}\tilde{y}_{t-1}^* + \phi_{22}\xi_{1t-1} + \phi_{23}\xi_{2t-1} + g_{21}\varepsilon_{y_t^*} + g_{22}\varepsilon_{yt} + g_{23}\varepsilon_{ct}.
\end{aligned}$$

[23] To see this, suppose for simplicity that we manipulate the error-correction terms so that they are relative to domestic GDP, rather than to technology. This can be done for all of the variables except domestic GDP itself, so it would remain relative to technology and therefore partially observed.

Consequently, in terms of the observable Δy_t,

$$\begin{aligned}\Delta y_t &= \Delta\tilde{y}_t + \varepsilon_t^a \\ &= \phi_{21}\tilde{y}^*_{t-1} + \phi_{22}\xi_{1t-1} + \phi_{23}\xi_{2t-1} + g_{21}\varepsilon_{y_t^*} + g_{22}\varepsilon_{yt} + g_{23}\varepsilon_{ct} + \varepsilon_{at} \quad (15)\end{aligned}$$

This is an identity. There will be a similar equation for $\tilde{c}_t$. The advantage is the separation of the EC terms into those that are observable - ξ_{1t}, ξ_{2t} - and only one that is partially unobservable - $\tilde{y}^*_t$.

This approach can also be applied to the MSM. Because it defines $\tilde{y}^{va}_t = y^{va}_t - a_t$, EC terms would be formed as $\xi^{va}_t = y^{va}_t - y^*_t$, $\xi^c_t = \tilde{c}_t - y^{va}_t = c_t - y^{va}_t$, $\xi^i_t = i_t - y^{va}_t$ etc., so that the domestic variables are relative to GDP while aggregate GDP is relative to foreign GDP. Ultimately, this means that there will then be only *one* partially unobservable EC term $(y^*_t - a_t)$. Fully unobserved variables such as a sectoral capital stock, $\tilde{k}_{m,u}$, are left in this form, as there is no point in expressing them relative to an observed variable. The representation for domestic output growth in MSM equivalent to (20) is presented in the Appendix.

What can we learn from this representation? First, we can gauge, among the many factors influencing GDP growth, the importance of the strictly unobservable variables.[24] If these are omitted from the regression, the R^2 goes from unity (recall this representation is an identity) to .998, so these contribute little to the explanation of GDP growth. Second, we can also look at the importance of the innovations to the shocks. Indeed, when all innovations ε_t are deleted the R^2 drops to .22. Hence *current shocks* are the most important factors affecting GDP growth. It is this fact that explains why recessions are so hard to predict, as future shocks must be known in order to predict whether future growth rates are negative. Essentially, this is an informal way to judge the likely performance of a model at forecasting.

It is also possible to examine which of these shocks is the most important. One might expect growth in non-stationary aggregate productivity, μ_t, to be that, but deleting only that shock reduces the R^2 from unity to .99. Deleting the separate domestic industry productivity shocks $\varepsilon_{a_{nt}}$ etc. has a much greater impact, with the R^2 going from unity to .82. But by far the most important single shock is the marginal efficiency of investment ε_{Υ_t}, since removing it reduces the R^2 to .71.[25] The R^2 available from the SSVAR, in this case reformulated using error-correction terms due to the presence of a permanent shock, is a useful metric for looking at either the importance of unobserved variables or the innovations.[26]

The analysis above can be repeated for inflation. Removing the innovations results in a substantial drop in the R^2 to .23. Again this suggests that it will be difficult to predict inflation. In contrast to the results for output growth,

[24] The unobserved variables are π^f_{t-1}, $\tilde{k}_{m,t-1}$, b^*_{t-1}, b^*_{t-2}, $\tilde{k}_{m,t-2}$, $\tilde{k}_{z,t-1}$, $\tilde{k}_{z,t-2}$, $\lambda_{z,t-1}$, $\lambda_{n,t-1}$ and $\lambda_{m,t-1}$.

[25] One can see this effect as well from Rees et al.'s figure 10.

[26] Of course when one turns to data rather than the hypothetical scenario the shocks are actually correlated, so that it is not possible to uniquely attribute any part of actual GDP movements to particular shocks.

deleting the unobservable variables from the regression results in an R^2 of .55. So this points to a problem of matching data and model variables. Indeed the first order serial correlation of inflation from the model is .34 and the data is .48.

There are other comparisons we might make, such as comparing the statistics on the observable EC terms in the model to the data. This could involve tests for whether the estimated EC terms are indeed $I(0)$. Instead, Table 5 presents some evidence of their volatility in the MSM model relative to the data. Doing so, it is apparent that the model generally produces much greater volatility than that evident in the data. With exactly identified models the estimated model variances would match those of the data.

Table 5: **Volatility of Selected Error-Correction Terms from the Estimated MSM Model**

Error-Correction	**Mnemonic**	**Standard Deviation**	
		Model	Data
Non-Traded GDP	ξ_t^n	1.17	0.79
GDP	ξ_t^{va}	1.99	1.72
Investment	ξ_t^i	7.66	4.97
Government Expenditure	ξ_t^g	5.12	2.60
Consumption	ξ_t^c	2.75	2.08

4 Identification Issues in DSGE Models

Multiple models with an equivalent fit may alternatively occur if some of the DSGE parameters, θ, are not well identified. In DSGE models identification issues can be a reflection of the model solution being largely invariant to different values of the elements of θ, or the likelihood being insensitive to the solution (see Iskrev 2010). In this section we discuss two methods that have been applied in the literature to ascertain if weak identification exists, practical issues that arose in their application to MSM, and how these can be handled.[27]

Consider the log likelihood $L(\theta)$ as a function of parameters θ. Then a second-order approximation around the maximum likelihood estimate, $\hat{\theta}$ yields

$$L(\theta) = L(\hat{\theta}) + \frac{1}{2}(\theta - \hat{\theta})' H_{\theta\theta}(\hat{\theta})(\theta - \hat{\theta}) + "terms", \tag{16}$$

where $H_{\theta\theta}$ is the Hessian $\frac{\partial^2 L}{\partial\theta\partial\theta'}$. The omitted "terms" should be smaller than the other elements. Hence

$$\frac{1}{T}\{L(\theta) - L(\hat{\theta})\} \approx \frac{1}{2}(\theta - \hat{\theta})'(T^{-1} H_{\theta\theta}(\hat{\theta}))(\theta - \hat{\theta}). \tag{17}$$

Now the $\lim_{T\to\infty} -\frac{1}{T}E[H_{\theta\theta}(\theta)] = I_{\theta\theta}$, the asymptotic information matrix and we might replace the above with

[27] Canova and Sala (2009) demonstrate that simply comparing the prior and posterior may not detect identification issues.

$$\frac{1}{T}\{L(\theta) - L(\hat{\theta})\} = -\frac{1}{2}(\theta - \hat{\theta})' I_{\theta\theta}(\hat{\theta})(\theta - \hat{\theta}) + o_p(1).$$

So then the magnitude of the right hand side indicates how much a change from $\hat{\theta}$ to another value θ would change the log likelihood (scaled by T). If we put $\theta = 0$ then $\sqrt{\hat{\theta}_i^2 I_{ii}(\hat{\theta})/2}$ might be used as an index of this for the $i'th$ parameter θ. If this is low then the likelihood does not change much when θ_i values depart from zero, a characteristic of weak identification.

The quantity $\ln(\sqrt{\hat{\theta}_i^2 I_{ii}(\hat{\theta})})$ is the "sensitivity" index that Dynare produces to assess the identification of the parameters - see Ratto and Iskrev (2010) - except that $\hat{\theta}_i$ in their case is not the MLE but rather the prior mean. A difficulty with the latter choice is that, while $\frac{\partial L}{\partial \theta}(\hat{\theta}) = 0$, this will not be zero at the prior mean, and so (17) has another term in it. If one is analyzing a DSGE model that has been estimated with Bayesian methods then the Bayes posterior mode would be a more appropriate choice.[28] Fundamentally, this is a scaling issue. Another problem is that we probably should evaluate the information matrix at $\hat{\theta}$ and not at a prior mean. Again, using the Bayes mode makes sense in a Bayesian context.

Although there is no threshold value of $\sqrt{\hat{\theta}_i^2 I_{ii}(\hat{\theta})/2}$ that might signal weak identification, the relative magnitudes provide a guide to which of the parameters are likely to be weakly identified. Using a value of $\sqrt{\hat{\theta}_i^2 I_{ii}(\hat{\theta})/2}$ of less than 1.7 we find that a number of parameters are potentially weakly identified, in particular the response of all inflation rates to marginal cost pressures (the "slopes" of the Phillips curves).[29] This suggests that one should examine more closely the estimated slopes of the Phillips curves.

Reverting back to Equation (**??**), we might have used $T^{-1}H_{\theta\theta}(\hat{\theta})$ as our criterion. Koop et al. (2013) suggested a "learning rate indicator" for identification which involves simulating the DSGE model with the estimated parameters, and then studying the rate at which the precision $H_{\theta\theta}(\hat{\theta})$ changes. They argue that, when there is more than a single parameter, this is a better check of identification than simply looking at the closeness of the posterior and prior for any single parameter and observe that, in an identified model, $H_{\theta\theta}(\hat{\theta})$ should rise at rate T, meaning that $T^{-1}H_{\theta\theta}(\hat{\theta})$ will tend to a constant. In contrast, if the parameter is weakly identified, it will rise at a slower rate, and so $T^{-1}H_{\theta\theta}(\hat{\theta})$ will decline. Hence, by simulating the model and then estimating $H_{\theta\theta}$ for a range of T we can determine whether there might be a weak identification problem.[30]

[28] This is because the mode is the θ that maximizes $C(\theta) = L(\theta) + \log p(\theta)$, where L is the log likelihood and $p(\theta)$ is a prior density. Consequently the mode sets $\frac{\partial C}{\partial \theta} = 0$ and we could apply the same expansion as above to $C(\theta)$ rather than $L(\theta)$. The negative of the expected value of the second derivatives of $\frac{1}{T}C(\theta)$ will be asymptotically the information matrix, since the prior is dominated as the sample size grows.

[29] In contrast, the parameter ρ_{r*}, which is the inertia effect in the foreign interest rate rule, has the sensitivity index at 76.

[30] The logic of this last remark comes from noting that $H_{\theta\theta}(\theta) = \sum_{t=1}^{T} \frac{\partial^2 L_t}{\partial\theta\partial\theta'}$ and so

In practice, a weakly identified model will become identified as T becomes very large, so that the index stops declining with very large T, which can be seen in Koop et al.'s tables.

The implementation of the test is not entirely straight-forward for MSM since the simulated $H_{\theta\theta}$ was sometimes not negative definite.[31] Consequently, we use a variant, focussing on identification issues for each parameter individually. Data is simulated from the MSM using the parameters estimated by Rees et al.. i.e. these are treated as the correct ones. Then a particular parameter is selected, call it θ_1, and it is estimated by maximum likelihood and Bayesian methods, *with all other parameters not being estimated but set to the true values.* Designating the computed standard deviation of this estimated coefficient by $\hat{\sigma}_1$, studying the rate of convergence of $\hat{\sigma}_1$ should tell us about identification. Koop et al. referred to this in their work on the New Keynesian Phillips Curve and they noted that an identified parameter had $\hat{\sigma}_1$ tending to zero much faster than for the unidentified ones. This approach circumvents the problem above and, *as only one parameter is being estimated*, it is quick to implement, even with large data samples.[32]

Estimation was performed on a simulated sample of $M = 1500$ data points using sub-samples rising from 100 to 1500 observations by 100 at a time.[33] It is useful to assess the speed of convergence by regressing the resulting estimated standard deviations for the single parameter being estimated against the sample size. Specifically, we regress $\ln \hat{\sigma}_1$ against a constant and $\ln(M)$. The coefficient on $\ln M, \hat{\gamma}$, should be -.5 for an identified parameter. In fact, the estimates for $\hat{\gamma}$ for the Phillips curve slopes are values between -.63 (non-resource exports) and -.91 (foreign economy), so this does suggest some weak identification issues. By comparison a parameter which has a very high sensitivity index, namely ρ_{r^*}, gives an estimate of -.56, with a standard deviation of .02.

A related criterion which uses the same simulated data is to study how the recursive estimate $\hat{\theta}_1$ constructed using the simulated data changes as the sample size grows. Since all parameters are set at the true values when simulating the data, and only θ_1 is estimated, we would expect that $\hat{\theta}_1$ should converge to its true value if θ_1 was strongly identified. Slow convergence points to there being

$E[H_{\theta\theta}(\theta)] = \sum E(\frac{\partial^2 L_t}{\partial\theta\partial\theta'}) = TE(\frac{\partial^2 L_t}{\partial\theta\partial\theta'})$ under stationarity i.e. $T^{-1}H_{\theta\theta}(\theta)$ should converge to a constant as $T \to \infty$. In analyzing weak identification the local to zero approach puts $E(\frac{\partial^2 L_t}{\partial\theta\partial\theta'}) = \frac{D}{\sqrt{T}}$. Consequently, $E[H_{\theta\theta}(\theta)] = \frac{TD}{\sqrt{T}} = \sqrt{T}D$, and so it grows at a slower rate than T. Under such conditions, when there is weak identification $\frac{1}{T}H_{\theta\theta}(\theta)$ will decline to zero as T increases.

[31] The problem was also encountered by Caglar et al (2011) in their application of the Koop et al. approach to the Smets and Wouters (2007) model. Their strategy to address this was to simply vary the sample until they got a $H_{\theta\theta}$ that was negative definite.

[32] As a referee observed, even if a single parameter was found to be strongly identified, it does not mean combinations of them are. So, if we find found that each of the individual parameters were strongly identified when the others are fixed, it would not be possible to conclude that all the model parameters are jointly identified.

[33] In other words, we are essentially performing a recursive estimation using samples 1-100,1-200, 1-300..... We actually simulated 10000 observations and dropped the first 8500 to eliminate any initial condition effects.

weak identification. Moreover, it is often the case that with weak identification one sees "jumps" in the estimated $\hat{\theta}_1$ as the sample size gets larger.

Figure 1 shows a plot of the estimated Phillips curve slope for the non-traded sector, κ_{π_n}, as the sample is expanded from 100 to 1500. The true value is .2902. Two estimators are given - unconstrained maximum likelihood and the Bayesian mode using the MSM priors. As expected, we see that there is very little difference between all three estimators as the sample size grows, since the prior gets dominated. Even in small samples the differences are not great. It should be noted that there is very little evidence of convergence to the true value of .29. This contrasts with what one sees for ρ_{r*}, where the true value is .928 and the estimated quantities start at .918 (for 100 observations) and finish at .93 (for 1500). We note that there is a sample where the estimated parameter for κ_{π_n} dropped to a very small value and, as mentioned above, that type of behaviour is consistent with weak identification. This pattern is repeated for the slope coefficients of all the Phillips curves. A difficulty in identifying the slope of the Phillips curves is not isolated to MSM; for a discussion see Schorfheide (2008).

In summary, if the data had been generated with the parameter values estimated by Rees et al. there would be a bias in some of the Phillips curves slopes.[34] Moreover it is clear that we could get values that are close to zero. This is despite the fact that the prior on κ_{π_n} that Rees et al.. used had a mean of 50. As the slope of the Phillips curve is a crucial parameter for policy assessments and forecasting it would seem that one would need to look at a wide range of parameter values for the slopes when conducting policy assessments, since the weak identification analysis suggests that they are very hard to estimate.

5 Comparing the Model and Data Generating Processes

It emerged from the VECM representation of MSM constructed above that the variance of GDP growth in the model considerably exceeds that in the data used in estimation.[35] What are the implications of this?

Suppose that we had a model with two shocks and one of the variables in the model was generated by

$$y_{1t}^M = d_1\varepsilon_{1t} + d_2\varepsilon_{2t},$$

where the ε_{jt} are uncorrelated. Assuming that the same structure is used for estimation we would end up with (in large samples)

$$y_{1t}^D = d_1 u_{1t} + d_2 u_{2t},$$

[34] In the light of recent discussion about the decline in values for slopes of Phillips curves it is interesting to note that the bias would be downward.

[35] This is also evident from Table 5 in Rees et al.

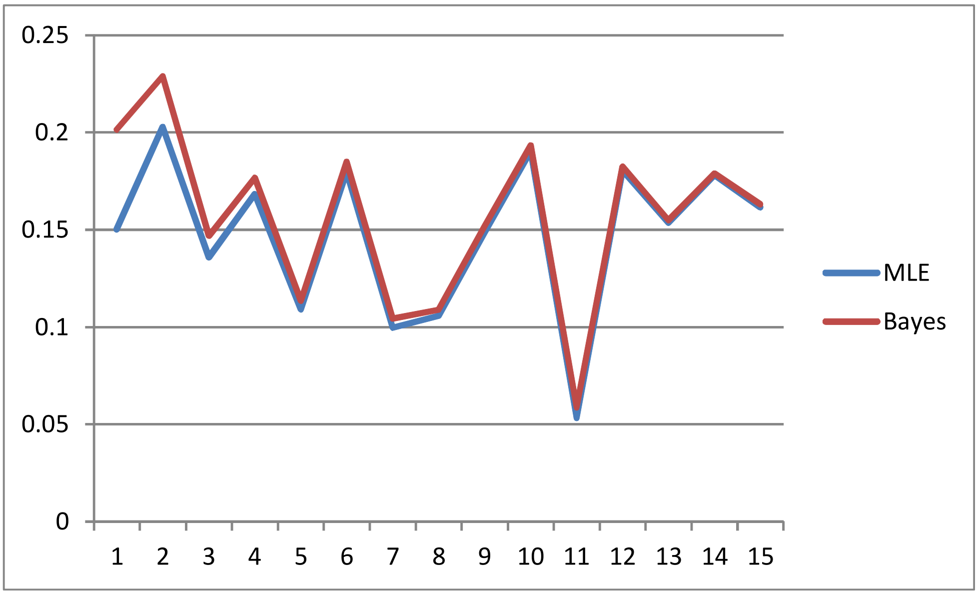

Figure 1: Recursive MLE and Bayesian Estimates of the Slope of the Phillips Curve for the Non-traded sector

where the u_{jt} are the shocks found after estimation. In large samples the variances of u_{jt} will equal that of ε_{jt}. Then, if $var(y_{1t}^D) < var(y_{1t}^M)$, it must be that there is a negative correlation between u_{1t} and u_{2t}. If it was positive then $var(y_{1t}^D) > var(y_{1t}^M)$. So this explains why the negative correlations between shocks found earlier can occur, and it reflects the fact that one of the assumptions used in estimation is incorrect. One possible response to this is to argue that the data has measurement error in it, and that is why the model variances do not match the data variances. The MSM model does incorporate such a feature, so we look at the issues of bridging data and model via measurement error in Section 5.1. Then in Section 5.2 we ask whether the presence of unobservable variables in the MSM would mean that we could not easily capture the impulse responses of the model by just using observable data. Finally, Section 5.3 asks whether the business cycles that would be produced by the MSM would resemble those of the Australian economy.

5.1 Bridging Data and Model via Measurement Error

One development in estimating DSGE models has been to build a bridge to the data via measurement errors. That is, if the model variable is y_t^M and the data is y_t^D, the equation $y_t^D = y_t^M + \zeta_t$ is added to the system. The implications of including measurement error ζ_t were analysed in Pagan (2017); here we discuss the results for the MSM.

A ζ_t will exist that reconciles the data and model variables. Watson (1993) considered this and, as he noted, some assumption has to be made about the relationship of y_t^M and ζ_t, i.e. how do the reconciliation shocks ("measurement errors") and the model shocks interact? One specification is that they are uncorrelated, and that is the primary assumption used in the MSM.

There are also other decisions that need to be made. Two stand out. First, how does ζ_t evolve, i.e. what is its nature? Second, do we fix or estimate the parameters of the generating process for the ζ_t? The answer to the first of these questions given in the MSM was to assume that ζ_t are white noise processes that are uncorrelated with one another. This means that the only parameters involved in the ζ_t processes are the variances of the shocks, and they were set in the MSM to values that were connected to the magnitude of y_t^D. The motivation for this approach seems to be that the model shocks would explain a certain percentage of the data while the "measurement error" accounted for the rest.

5.1.1 Parameter Choices for the Measurement Error Process

To examine the consequences of fixing the variance of the shocks ζ_t, particularly for multi-sector models like the MSM, we use the fact that aggregate GDP growth in the model is constructed by weighting sectoral GDP growth rates. This follows from Equation A27 of Rees et al. (2016) and it will yield[36]

$$\Delta y_t^{va} = \omega_1 \Delta y_{nt}^{va} + \omega_2 \Delta y_{mt}^{va} + \omega_3 \Delta y_{zt}^{va}. \tag{18}$$

All observed growth rates differ from the model equivalents according to some measurement errors (here "*obs*" indicates the observed data):

$$\begin{aligned}
\Delta y_t^{va,obs} &= \Delta y_t^{va} + \zeta_t^{va} \\
\Delta y_{nt}^{va,obs} &= \Delta y_{nt}^{va} + \zeta_{nt}^{va} \\
\Delta y_{mt}^{va,obs} &= \Delta y_{mt}^{va} + \zeta_{mt}^{va} \\
\Delta y_{zt}^{va,obs} &= \Delta y_{zt}^{va} + \zeta_{zt}^{va}.
\end{aligned}$$

Using (18) we have that the difference between observed GDP growth Δy_t^{va} and the weighted average of the observed sectoral growth rates is $\psi_t = \zeta_t^{va} - \omega_n \zeta_{nt}^{va} - \omega_m \zeta_{mt}^{va} - \omega_z \zeta_{zt}^{va}$. Now, from the data the standard deviation of ψ_t is .49, while using the parameter values from Rees et al.. of $\omega_n = .64$; $\omega_m = .23$,$w_z = .13$, $\sigma_n = .18, \sigma_m = .36, \sigma_z = .74$, $std(\eta_t^{va}) = \sigma_y^{va} = .18$, we would get a value of .25. Accordingly, it is clear that the measurement error shocks used in MSM do not provide a reconciliation of the model with the data.

How this can be so? One possible reason is that the standard deviations of the measurement errors are being set rather than being estimated. However, estimating these does not substantially change the result.[37] Other possible reasons relate to the other assumptions being made. One assumption is that the

[36] The sectors are non traded (n), non-resource exportables (m) and resource exportables (z).

[37] One further aspect to note is that, unless these standard deviations are parameters to

measurement errors are uncorrelated with the model variables. In fact, this is not the case; e.g. the correlation between the technology shock and measurement error in resource exports is .4, that between the risk premium shock and the nominal exchange rate measurement error is .83. Another assumption is that the measurement error shocks are uncorrelated with each other. Again, there are actually many substantial correlations; for example those between η_t^{va} and the three industry GDP measurement error shocks η_{nt}^{va} $(-.79)$, η_{mt}^{va} $(-.82)$ and η_{zt}^{va} $(-.74)$, while the correlation between measurement errors in investment and consumption is .7.[38]

5.1.2 The Interrelationship of Measurement Error Specification and Data

The nature of the ζ_t needed to reconcile the data and model variables may also be an issue. The only variables exempt from measurement error in the MSM are the nominal interest rates. To see the possible unintended consequences of including measurement error shocks we look at the change in the real exchange rate (q_t), which is defined as (from their Equation A31)

$$\Delta q_t = \Delta s_t + \pi_t - \pi_t^*,$$

where s_t is the log of the nominal exchange rate. Then the data variables are $\pi_t^D = \pi_t^M + \zeta_t^\pi, \pi_t^{*D} = \pi_t^{*M} + \zeta_t^{\pi^*}$, where the ζ_t are measurement errors. Hence we have for the data

$$\begin{aligned} \Delta q_t^D &= \Delta s_t^D + \pi_t^D - \pi_t^{*D} \\ &= \Delta s_t^M + \zeta_t^{\Delta s} + \pi_t^M - \pi_t^{*M} + \zeta_t^\pi - \zeta_t^{\pi^*}. \end{aligned}$$

Cumulating these produces

$$q_t^D = s_t^M + P_t^M - P_t^{*M} + \sum_{j=1}^{t} (\zeta_j^{\Delta s} + \zeta_j^\pi - \zeta_j^{\pi^*}).$$

In the model $q_t^M = s_t + P_t^M - P_t^{*M}$ is an I(0) process, since there is co-integration between the nominal exchange rate and the relative prices. Because $\sum_{j=1}^{t} (\zeta_j^{\Delta s} + \zeta_j^\pi - \zeta_j^{\pi^*})$ is an I(1) process it follows that, unless the variance of

estimate, observed GDP is redundant. To see this, recall that the observed data for MSM consists of seventeen variables. These include the aggregate GDP growth Δy_t^{va} and the sectoral ones Δy_{jt}^{va} $(j = m, n, z)$. Given Δy_{jt}^{va} and the other 13 variables one can set up a likelihood to find estimates of the MSM parameters θ. Defining $\Delta y_t^S = \sum_j \omega_j \Delta y_{jt}^{va}$, where ω_j are weights used in the MSM, then we have $\Delta y_t^{va} = \Delta y_t^S + \zeta_t^{va}$, where ζ_t^{va} is a reconciliation or measurement error. So Δy_t^{va} is not used in estimating θ but would only be used to find the variance of ζ_t^{va}. Hence it means that there are only 16 observable variables being used to estimate the 17 MSM shocks and so these 17 cannot be uniquely recovered. This also meant that in the weak identification analysis above, which was conducted without measurement error, observed GDP growth was omitted to avoid a singularity.

[38] Because the DSGE model is over-identified we can estimate these correlations.

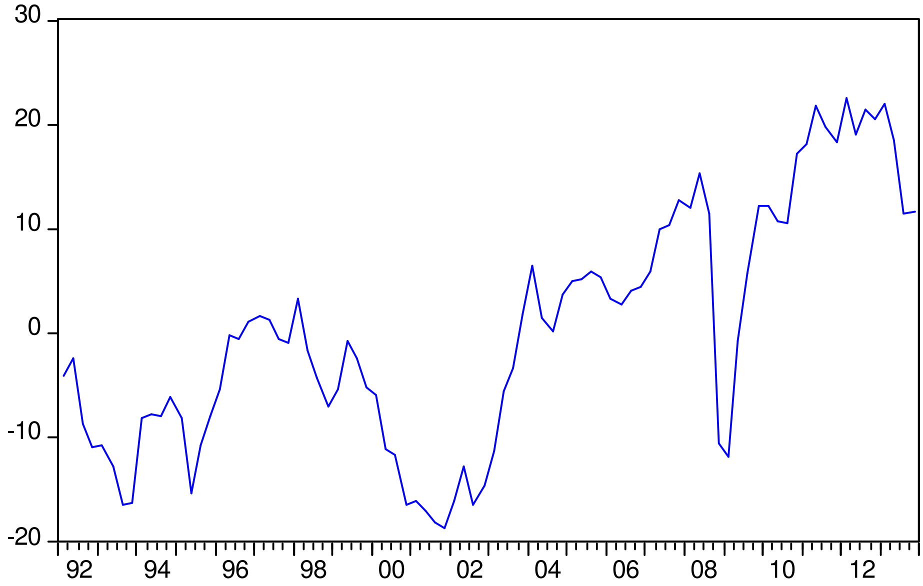

Figure 2: Log of the Real Exchange Rate

$(\zeta_j^{\Delta s} + \zeta_j^{\pi} - \zeta_j^{\pi^*})$ is zero, the real exchange rate *in the data* is predicted to be $I(1)$. Because in the MSM measurement errors are taken to be independent and white noise processes the variance of $(\zeta_j^{\Delta s} + \zeta_j^{\pi} - \zeta_j^{\pi^*})$ will not be zero. In other words the introduction of measurement errors predicts a lack of co-integration between the nominal exchange rate and relative prices in the data.

Applying an ADF test to the latter series (based on four lags) gives -1.58, which does seem to point to an I(1) process, and implying that this could be handled by allowing for measurement errors in the domestic and foreign inflation rates. Looking at the plot of the real exchange rate in Figure 2 (this is after mean correcting the change in the nominal exchange rate and the domestic and foreign inflation rates) it is apparent that there is a deterministic trend in the data i.e. a shift in the mean of the growth rates after 2003 rather then a stochastic trend. Kulish and Rees (2015) noted this and proposed that one allow for such trend shifts in a model like MSM. So the incorporation of white noise measurement errors does not provide a satisfactory solution. One needs to effect any reconciliation with a different method. Relatedly, if a model is to be used for policy analysis one needs to make some assumptions about the nature of the reconciliation shock into the future, as the policy maker is ultimately interested in the implications for the actual data, rather than the model variables.

Similar implications extend to many other variables in the model. A noteworthy example is that y_t^{va} and y_t^* in the model co-integrate but they will be predicted to not have that property in the data. Clearly measurement error could be a useful device to reconcile data and model variables when there is co-integration implied by the model but it is not present in the data. Viewed in this way ζ_t is best thought of as a reconciliation shock, since it aims to reconcile the model and data, rather than measurement error. In order for the reconciliation shocks, ζ_t to actually achieve a reconciliation between the model and data the way they are modelled is important. For example, if none of the error-correction terms defined by the model are $I(0)$ in the data, then one can choose ζ_t to be white noise. However, if a model-defined error-correction term is $I(0)$ in the data, an assumption of white noise measurement errors is incorrect, since the data and model are already reconciled on that dimension.[39]

There is an argument for re-structuring DSGE models like MSM as we did for the SSVAR, expressing them in terms of EC terms plus $y_t^* - a_t$. Then we could work with the growth rates, error-correction terms and Δy_t^* as the data, and have the reconciliation shocks placed on the error correction terms. That enables more flexibility than placing them on the growth rate data.

5.2 Approximating DSGE Models with an Observable Variables SSVAR

As we saw earlier in section 2.1 most DSGE models have an SSVAR involving both observable and unobservable variables. So what happens if one only fits an SSVAR with observable variables? This is of interest because, if it is found that the DSGE model can be approximated well by a SSVAR(2), then it suggests one could obtain similar results from other models as long as they also have such a representation. There could be advantages to working with these other models, in particular institutional features of the economy under investigation may be captured more easily with them..

Pagan and Robinson (2016) looked at this for a range of DSGE models in the literature and found that, in many cases, an SSVAR(2) fitted quite well, in the sense of being able to generate impulse responses that were those of the underlying DSGE model. It is always worth looking at this after a DSGE model has been constructed. To illustrate take the $\tilde{y}_t^*$ of the external sector of MSM and set up its connection with observed foreign GDP growth Δy_t^*

$$\Delta y_t^* = \Delta \tilde{y}_t^* + \Delta a_t = \Delta \tilde{y}_t^* + \mu_t. \tag{19}$$

Equation (11) provides an expression for $\tilde{y}_t^*$ from the solution to the MSM and, combining this with (19), we get

$$\Delta y_t^* = -.248\pi_{t-1}^* + .083r_{t-1}^* + \{-.135\tilde{y}_{t-1}^* + .003\varepsilon_{yt}^* - .005\varepsilon_{\pi t}^* - .049\varepsilon_{rt}^* + \mu_t\}.$$

[39] A referee asked whether one might treat the measurement error as being on the $I(1)$ variable. If that error is taken to be $I(0)$ then it will disappear relative to the variance of the "true" I(1) variable as the sample size grows. Pagan(2017) discusses methods that one might use to avoid the lack of co-integation issues raised here. One is to assume that the measurement error in growth rates is $\Delta\zeta_t$, where ζ_t is white noise, and this is equivalent to adding ζ_t on to the level of the model variables. A more appealing solution though is to introduce an error correction term involving the measurement error.

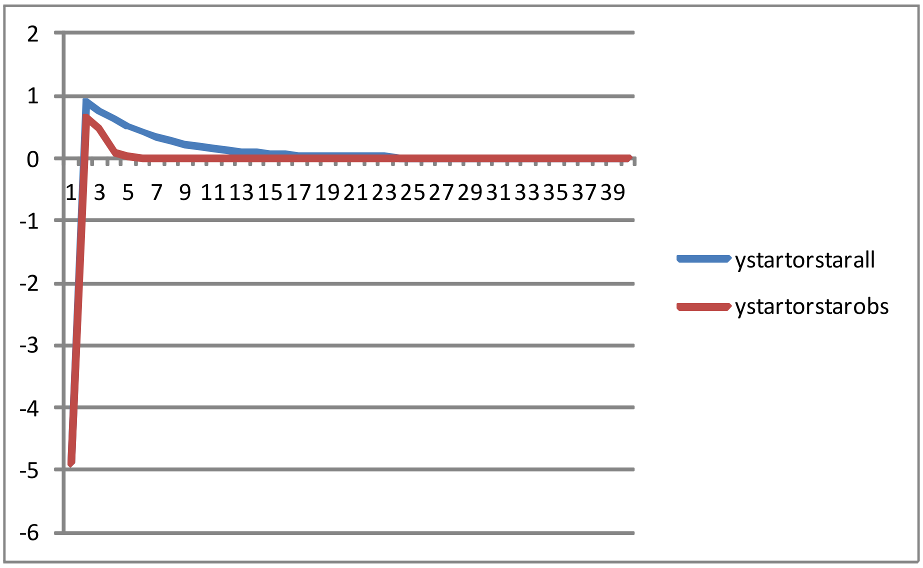

Figure 3: Impulse Response of Foreign GDP Growth to a Foreign Monetary Policy Shock, All and Just Observed Variables

If the term in brackets is close to white noise then a VAR(1) in $\Delta y_t^*, \pi_t^*$ and r_t^* would fit this equation quite well. Essentially the question of how important the approximation error is boils down to comparing the relative variances of the omitted term $-.135\tilde{y}_{t-1}^*$ and the disturbance $.003\varepsilon_{yt}^* - .005\varepsilon_{\pi t}^* - .049\varepsilon_{rt}^* + \mu_t$. Since the $var(\tilde{y}_t^*) = .0001$ and $var(\mu_t) = .0001$, it follows that the series in brackets will look like white noise. Indeed if one fits a MA(1) process to it one finds a coefficient of .07. The situation differs for the π_t^* equation, where the MA(1) coefficient is .24. So we might expect some difficulties in capturing the inflation responses. Figures 3 and 3 below show that this seems to be true; in these figures we look at the ability of an SSVAR(2) in observables $\Delta y_t^*, \pi_t^*$ and r_t^* to capture the monetary impulse responses for the foreign sector of the MSM. Note that in order to study the pure approximation error we begin the impulse responses with the values of the contemporaneous responses - the argument for this is given in Pagan and Robinson (2016).

We can also study impulse responses for the domestic sector. Liu et al. (2018) give a variety of these and we show two here in Figures 5 and 6 - inflation and the real exchange rate responses to a monetary shock. Generally, the correspondence is quite good for all impulse responses, particularly over the first two years.

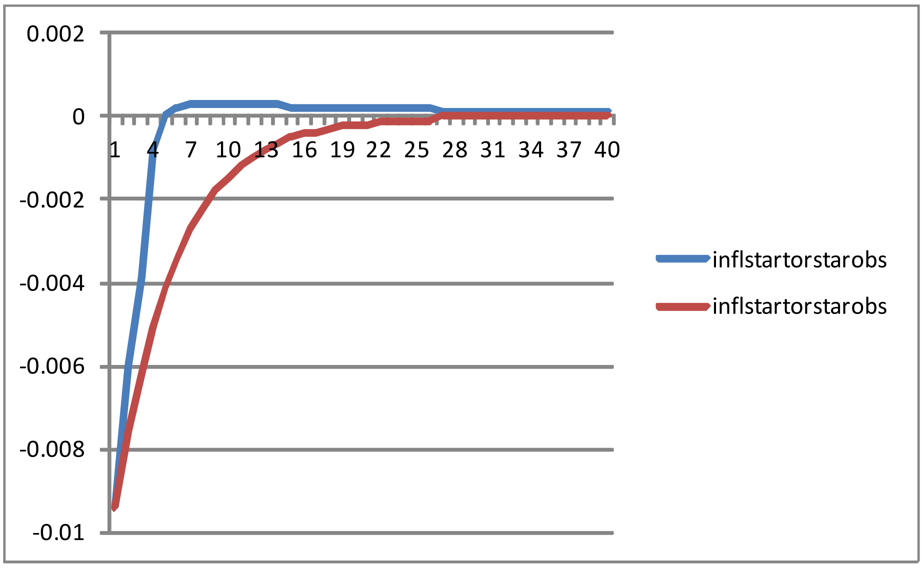

Figure 4: Impulse Response of Foreign Inflation to a Foreign Monetary Policy Shock, All and Just Observed Variables

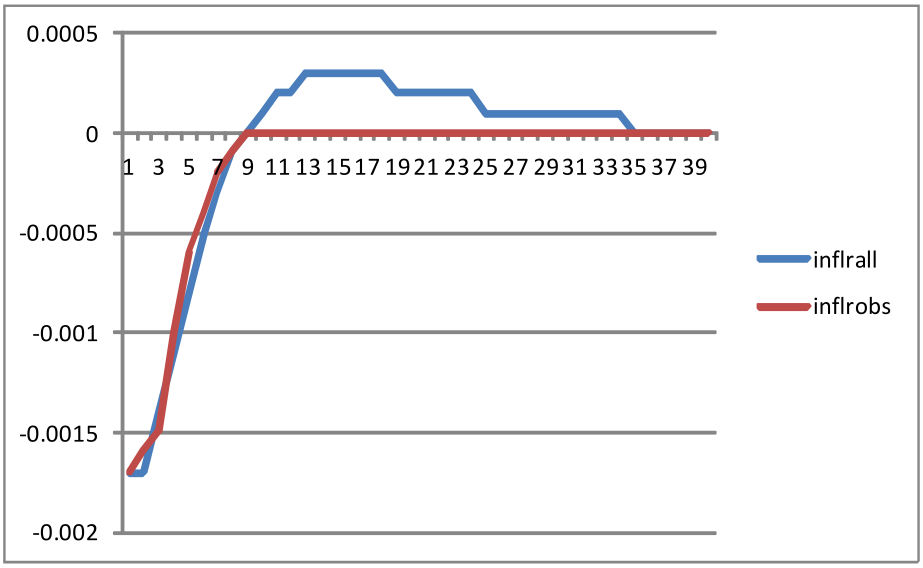

Figure 5: Impulse Response of Domestic Inflation to a Domestic Monetary Policy Shock, All and Just Observed Variables

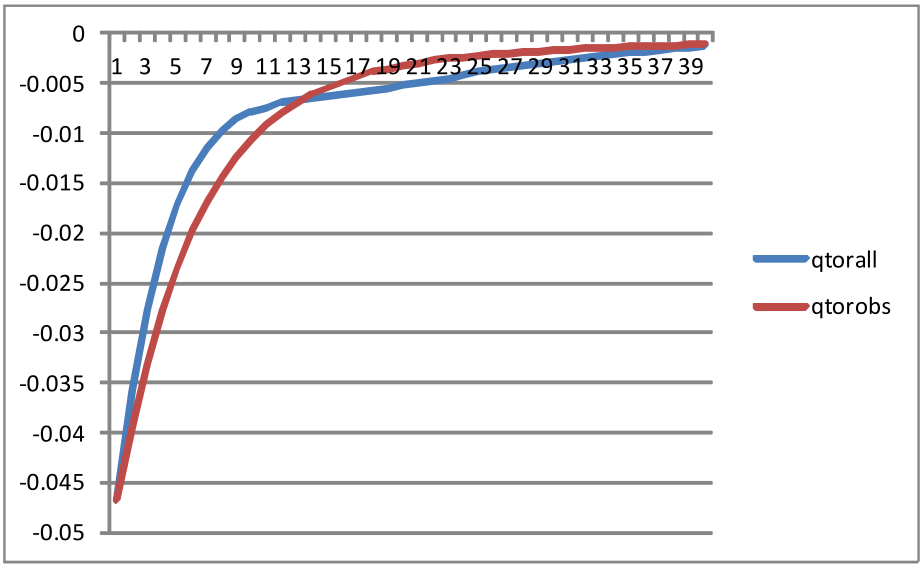

Figure 6: Impulse Response of the Real Exchange Rate to a Domestic Monetary Shock, All and Just Observed Variables

5.3 Examining Some Features of the DSGE Generating Process in Relation to Data

It is often not realized that there is a need for a DSGE model to replicate the volatility of GDP growth if it is to produce historical business cycle outcomes. To illustrate this we look at the business cycle outcomes embedded in the MSM since we know that it produces an excessive volatility in GDP growth. One outcome that is always worth looking at is the number of negative growth rates we would get from the model versus that in the data. To examine that we add back the mean growth rate to the model series and then ask how many negative growth rates would be produced. For the MSM we find that there are four times as many as in the data. This suggests that the model will produce quite a lot of recessions since any recession starts with a negative growth rate. Alternatively, one can identify recessions using the turning points in the level of the series, found with rules like those used by the NBER to date business cycles. This literature is surveyed in Harding and Pagan (2016). Pagan and Robinson (2014) used it to assess financial DSGE models, finding that they were unable to produce realistic characteristics of business and financial cycles. So we simulate GDP data from the MSM model (with uncorrelated shocks) and apply the BBQ program of Harding and Pagan(2002) in order to locate turning points.[40] These are presented in Table 6.

[40] This is available as an add on to EViews and is present in other sources such as R. Matlab and Excel versions of BBQ are at http://www.ncer.edu.au/resources/data-and-code.php

Table 6 The MSM Business Cycle Characteristics					
			MSM (all shocks)	**MEI Omitted**	**Foreign Omitted**
Durations					
Dur Con			2.5	2.5	2.7
Dur Expan			27.3	54.6	28.7
Amplitudes					
Amp Con			-1.2	-.93	-1.2
Amp Expan			25.0	47.2	26.4

We see from this that the model produces a complete cycle on average every 7 years. This is considerably different to the recent Australian experience, as the current expansion has lasted more than twenty years. Applying the dating algorithm to the actual data used in the MSM estimation we also find that there were no recessions. Consequently, it is clear that the model has an inbuilt feature that would predict recessions too often.

Business cycle dating can also be used to ascertain the importance of particular shocks in determining the business cycle characteristics. This is simple to implement, in that it involves simulating data from the model with that shock turned off, applying the dating algorithm, and then comparing the cycle characteristics to those obtained with all of the shocks. Such exercises are shown in Table 6. Table 6 shows that the characteristics are dramatically altered when the marginal efficiency of investment shock is omitted - in particular, the duration and amplitude of expansions approximately double. Using different methods Justiniano et.al. (2011) found this shock to be important in U.S. models, and they argue that it is proxying for financial shocks. In contrast to the MEI shocks, Table 6 shows that omitting external shocks has relatively little impact.[41]

6 Conclusion

The construction of large DSGE models are impressive achievements. This is very true of the MSM. It provides a feasible way of implementing the tradeable/non-tradeable model of a small open economy, while at the same time handling a number of sectors that are an important institutional feature of the Australian economy.

In many instances DSGE models do not match some characteristics of the data. This may not be surprising; ultimately all empirical macroeconomic modelling involves making compromises along some dimensions. Our work, however,

[41] See Justiniano and Preston (2010), who show that small-open economy DSGE models often attribute a surprisingly small role to foreign disturbances.

suggests that a failure to match data on some levels can have broad implications for the nature of the shocks that are in the data, and these may not be adequately recognized in the estimation and use of the model. The MSM model (and many other DSGE models) are being used as if the shocks have exactly the same properties as were assumed about them in estimation. When DSGE models are exactly identified this is correct. However, DSGE models are typically over-identified and, consequently, experiments being performed under the estimation assumptions about the shocks are hypothetical, and may be contrary to the data. Such experiments may be suitable for understanding the ways in which shocks potentially work through an economy, but whether the data-compatible shocks enter in the same way is unknown.

The essential point is that once any DSGE model is established and estimated, one needs to spend time investigating its properties, particularly in relation to the the data. This involves more than just looking at a few moments or (in a Bayesian context) a marginal likelihood. Simulating the model so as to study other ways of judging adequacy, such as business cycle outcomes, often provides insight into weaknesses that are not apparent from studying a few moments. If there is a failure along these dimensions then the question that needs to be raised is whether something important has been missed. In addition to this the question of whether parameters are well identified should also be examined. In the case of MSM our analysis suggests that the slopes of the Phillips curves in all of the sectors may only be weakly identified. This means that when the model is used, one should assume a range of scenarios with different parameter values.

The practice of adding measurement error into DSGE models (which has become quite common) often leads to unintended implications that are assessable from the data. This problem is particularly acute when such errors are associated with growth rates in variables (as in the MSM), since then there are testable co-integration implications. This was set out in Pagan (2017), and here it was applied in the context of the MSM. Our feeling is that it is generally not a good idea to claim that there is measurement error. In the event that one does, it needs to be done with great care. In particular, one should check that the implications of doing so are valid.

Many DSGE models today, including MSM, include a unit root technology process. This implies that co-integration exists between many of the real variables. We re-formulated MSM as an SVAR with growth rates and error-correction terms. This had the advantage that one could focus on error-correction terms between observable variables rather than the unobserved technology level. Using this reformulation it was possible to easily compare the properties of error-correction terms in the model to those in the actual data.

Many DSGE models are used for policy analysis rather than forecasting and this is true of MSM. It was suggested that a way of reporting results from such a model which is potentially useful to policy makers is to present the range of impulse responses from models that provide the same fit to the data. This is a way to communicate the degree of model uncertainty that exists. It demonstrated, for example, that in MSM the contemporaneous response in the external sector to a monetary policy shock in MSM is very large compared to that from many equivalent models.

Finally, we looked at whether it would be possible to approximate the MSM impulse responses with a SVAR. Here the complication is that there are variables in the MSM that have no counterpart in the data used in its estimation, such as foreign debt and the capital stocks in each of the sectors. We found that an SVAR(2) in just the observable variables could do quite well in capturing the responses. This suggests that one might reproduce many of the MSM results with alternative types of models, provided they have a SVAR(2) representation. This might be appealing as it could be easier to incorporate institutional features of the Australian economy into these alternative models.

In summary, in this paper we have examined ways in which a DSGE model can be assessed after estimation, focussing particularly on the nature of shocks and what can be learnt from alternative representations of it.

7 References

Andrle, M. (2014) "Estimating Structural Shocks with DSGE Models" (mimeo)

Arias, J.E., J.F. Rubio-Ramírez and D.F. Waggoner (2014). "Inference Based on SVARs Identified with Sign and Zero Restrictions: Theory and Applications," *Dynare Working Papers 30*, CEPREMAP.

Baumeister, C. and J. D. Hamilton, (2015), "Sign Restrictions, Structural Vector Autoregressions, and Useful Prior Information," *Econometrica*, 83, 1963-1999.

Binder,M and M.H.. Pesaran (1995), "Multivariate Rational Expectations Models and Macroeconomic Modelling: A Review and Some New Results" in M.H. Pesaran and M Wickens (eds) *Handbook of Applied Econometrics: Macroeconomics,* Basil Blackwell, Oxford.

Caglar, E., J. S. Chadha & K. Shibayama, (2011), "Bayesian Estimation of DSGE models: Is the Workhorse Model Identified?," *Studies in Economics 1125*, School of Economics, University of Kent.

Canova, F. and G. De Nicoló (2002), "Monetary Disturbances Matter for Business Fluctuations in the G-7", *Journal of Monetary Economics*, 49, 1131-1159.

Canova, F. & L. Sala, (2009) "Back to square one: Identification issues in DSGE models," *Journal of Monetary Economics*, 56(4), 431-449.

Christensen, T.M., A.S. Hurn and A.R. Pagan (2011) , "Detecting Common Dynamics in Transitory Components", *Journal of Time Series Econometrics*, Vol 3/1, Paper 3.

Del Negro, M. & F. Schorfheide (2004) "Priors from General Equilibrium Models for VARS," *International Economic Review*, 45, 643-673.

Fry, R. and A.R. Pagan (2011) ," Sign Restrictions in Structural Vector Autoregressions: A Critical Review", *Journal of Economic Literature,* 49, 938-960.

Harding, D., and Pagan, A.R. (2002). "Dissecting the Cycle: A Methodological Investigation", *Journal of Monetary Economics*, 49, 365-381.

Harding, D. and A. Pagan (2016), *The Econometric Analysis of Recurrent Events in Macroeconomics and Finance* (Princeton University Press, Princeton)

Iskrev, N. (2010) "Local identification in DSGE models," *Journal of Monetary Economics*, 57, 89-202..

Justiniano, A. and B. Preston (2010) "Can structural small open-economy models account for the influence of foreign disturbances?,"*Journal of International Economics*, 81, 61-74.

Justiniano, A. and G. Primiceri and A. Tambalotti (2011) "Investment Shocks and the Relative Price of Investment," *Review of Economic Dynamics*, 14,101-121.

Koop, G., M.H. Pesaran and R.P. Smith (2013), "On Identification of Bayesian DSGE Models", *Journal of Business and Economic Statistics*, 31, 300-314.

Kulish, M. and D. Rees (2015), "Unprecedented Changes in the Terms of Trade", *Reserve Bank of Australia Discussion Paper* 2015-11

Liu X, Pagan AR, Robinson T, (2018), Critically assessing estimated DSGE models: A case study of a multi-sector model", *Working Paper 4/2018,* Centre for Applied Macroeconomic Analysis, Crawford School of Publkic Policy, Australian National University.

Metaxas, P.M. and E. J. Weber (2016), "An Australian Contribution to International Trade Theory: The Dependent Economy Model", *Economic Record*, 92, 464-497.

Ouliaris, S. and A. Pagan, (2016), "A Method for Working with Sign Restrictions in Structural Equation Modelling" *Oxford Bulletin of Economics and Statistics*, 78, 605-622

Pagan, A. (2017), " Some Consequences of Using "Measurement Error Shocks" When Estimating Time Series Models", *CAMA Research Paper,12/2017,* Australian National University

Pagan, A. and T. Robinson (2014), "Methods for Assessing the Impact of Financial Effects on Business Cycles in Macro-econometric Models" *Journal of Macroeconomics* , 41, 94 -106

Pagan, A.and T. Robinson (2016), "Investigating the Relationship Between DSGE and SVAR Models", *NCER Working Paper 112*, Queensland University of Technology

Ratto, M. and N. Iskrev (2010) "Identification toolbox for DYNARE. ", available at http://www.dynare.org/CEF2011/identification.pdf

Rees, D.M. Smith, P. and Hall, J. (2016), "A Multi-sector Model of the Australian Economy", *Economic Record*, 92, 374–408

Rubio-Ramirez, J.F., D.F. Waggoner and T. Zha (2010), "Structural Vector Autoregressions: Theory of Identification and Algorithms for Inference," *Review of Economic Studies,* 77, 665-696.

Schorfheide, F. (2008) "DSGE Model-based Estimation of the New Keynesian Phillips Curve," *Economic Quarterly,* Federal Reserve Bank of Richmond, Fall, 397-433.

Schorfheide, F. (2013) “Estimation and Evaluation of DSGE Models: Progress and Challenges", in D. Acemoglu, M. Arellano, and E. Deckel (eds.), *Advances in Economics and Econometrics*, Volume 3, 184–230, Cambridge University Press, Cambridge.

Smets, F and R. Wouters (2007), "Shocks and frictions in US business cycles: a Bayesian DSGE approach", *American Economic Review*, 97, 586-606.

Watson, M.W. (1993), "Measures of Fit for Calibrated Models," *Journal of Political Economy*, 101, 1011-1041.

8 Appendix

8.1 VECM Representation of Domestic GDP Growth in MSM

The expression for the growth in domestic GDP from the MSM model, omitting variables whose coefficient is zero, is

$$\begin{aligned}\Delta y_t^{va} \;=\; & .23\Delta y_{t-1}^{va} - .015\Delta y_{t-1}^{*} - .007\xi_{t-1}^{zxy} - .01\xi_{t-1}^{yxm} - .015\xi_{t-2}^{yxm} - \\ & .03\pi_{t-1} + .08\xi_{t-1}^{c} - .171\xi_{t-2}^{c} - .25\xi_{t-1}^{va} + .02q_{t-1} + .04p_{t-1}^{*z} - .088\xi_{t-1}^{van} \\ & +.11r_{t-1}^{*} + .033\xi_{t-1}^{vaz} - .02\pi_{t-1}^{n} - .41r_{t-1} - .008\pi_{t-1}^{*} + .006\xi_{t-1}^{i} \\ & -043\xi_{t-1}^{i} - .019\xi_{t-1}^{g}\{-.22\tilde{y}_{t-1}^{*} + .02\pi_{t-1}^{f} - .169b_{t-1}^{*} + .168b_{t-2}^{*} + \\ & 086\tilde{k}_{m,t-1} - .063\tilde{k}_{m,t-2} - .082\tilde{k}_{z,t-1} + .062\tilde{k}_{z,t-2} - .094\lambda_{z,t-1} + \\ & .041\lambda_{n,t-1} + .074\lambda_{m,t-1}\} - .01\varepsilon_{r,t} - .0033\varepsilon_{r_t^*} + .0005\varepsilon_{ft} - .0004\varepsilon_{mt} \\ & -.003\varepsilon_{nt} - .0015\varepsilon_{m_t^*} + .0004\varepsilon_{y_t^*} + .0004\varepsilon_{\pi_t^*} + .0009\varepsilon_{\Upsilon_t} + .0005\varepsilon_{a_{mt}} \\ & +.0007\varepsilon_{a_{nt}} - .59\mu_t + .001\varepsilon_{a_{zt}} + .0015\varepsilon_{gt} - .0007\varepsilon_{\psi t} + .0002\varepsilon_{p_t^*} + .0008\varepsilon_{\xi_{c,t}}\end{aligned} \tag{20}$$

The terms in the braces are the unobserved variables.

Investigating the Relationship Between Economy-Wide and SVAR Models*

Adrian Pagan† and Tim Robinson‡

April 27, 2018

Abstract

Economy-wide models, particularly DSGE models, often contain variables which are not observed in estimation. A finite-order VAR representation of the model may not exist in the observed variables alone. This creates a VAR-truncation problem and the impulse responses can diverge. This paper quantitatively examines the issue. It is found to often not be important, and to reflect the omission of stock variables from the VAR. Divergences in the impulse responses can occur absent the VAR-truncation problem due to differing contemporaneous responses. We demonstrate that DSGE models incorporate strong identifying restrictions which are seldom used to identify SVAR models.

1 Introduction

Comparing impulse responses from differing models is a common form of model validation. Structural Vector Autoregressions (SVARs) provide economic interpretations of the shocks hitting the economy, but have relatively loose theoretical underpinnings compared to other structural modelling approaches. Consequently SVAR models may be thought as potentially being more aligned with the data. Their impulse responses are often used as a check on whether more tightly-specified, larger, models are capturing the data adequately, e.g. Brayton, Laubach and Reifschneider (2014, p.4) say, in the context of the Federal Reserve's FRB/US model, that "The responses of the output gap and inflation to a permanent increase in multi-factor productivity are also in general accordance with estimates from the VAR literature of the effects of technology shocks".

But are such comparisons reasonable? Because a SVAR is relatively simple, and uses less theory, a natural question to ask is whether it is indeed possible for a SVAR to closely

*We thank participants at 2016 Padova Macro Talks, a seminar at the University of Tasmania, Efrem Castelnuovo and Mariano Kulish for comments on an earlier version of this paper. Research Supported by ARC Grant DP160102654.

†School of Economics, University of Sydney and Centre for Applied Macroeconomic Analysis, Australian National University. E-mail: adrian.pagan@sydney.edu.au

‡Melbourne Institute of Applied Economic and Social Research, University of Melbourne. E-mail: tim.robinson@unimelb.edu.au.

capture the responses of a model which represents the actual economy. We will refer to the latter as the Target Model (TM), as they constitute the target for judging the success of the SVAR. This question has been examined in the literature. For example, Chari, Keheo and McGrattan (2005) and Christiano, Eichenbaum and Vigusson (2007), using DSGE models as the TMs, reached conflicting conclusions; Chari et al. concluded that SVARs could not capture the impulse responses, whereas Christiano et al. were more optimistic. Kapetanios, Pagan and Scott (2007), using as the TM an economy-wide model that was a miniature version of that used in the Bank of England in the 2000s, found a very high-order SVAR to be necessary to precisely replicate TM's impulse response functions.

The research question we focus on is assessing the ability of SVARs to "make a match" with the TM's impulse response functions and to ascertain the properties of the TM that influence this. The strategy we adopt is to examine how close a match is made in a range of TMs drawn from the literature which vary considerably in their characteristics. For example, we consider models with and without nominal rigidities, closed and small-open economy models, and those with just transitory shocks or a mix of transitory and permanent shocks. This is different to much of the literature, which assesses the quality of the match for one, or at most two, types of models.

Traditionally in simultaneous equation systems a distinction was made between endogenous and exogenous variables. In their itemization of endogenous variables in the FRB/US model Brayton et al. (2014) refer to a "core set of variables". This distinction reflects that many of the endogenous variables may be derivative from the core set of endogenous variables, together with exogenous variables via identities. In FRB/US, for example, there are 50 core variables, but around 375 variables in total. Essentially, there is a subset of the variables which we focus on when analysing the properties of the model.

In estimation not all the core variables may be directly observed, that is, have a counterpart in the actual data. Many variables will instead be latent.[1] As an example, often the exogenous variables will be latent simply because they are modelled as a shock following a simple statistical process; this is typified by the treatment of world GDP in many small-open economy models.[2] When assessing how well a match can be made between the impulse response functions of the SVAR and the TM, we will focus on the responses of the observed variables.

There are two dimensions to making a match. The TMs considered are linearised, and their solution in terms of the core variables and exogenous innovations will be a SVAR(p). The impulse response at the $j'th$ horizon from the TM will reflect both its contemporaneous responses and the dynamics of the model (i.e. its VAR coefficients).[3] However, a researcher

[1]One might argue that a recent practice of adding "measurement errors" on to a model variable to match the data means that all variables are latent. There are difficulties with using measurement errors in TMs that have been set out in Pagan (2017). We will keep to the division of core endogenous variables into observable and latent variables by not allowing for measurement error.

[2]Latent variables, however, may be indirectly observable as the observed data and the model together can be used to produce an estimate of them. Examples would be either an output gap or a flexible-price (hereafter flex-price) equilibrium. For convenience, however, we will refer to the latent variables as unobservable since one needs a model to measure them and there are no *direct* observations on them.

[3]This reflects the fact that the $k'th$ period ahead impulse responses of z_t to ε_t from a SVAR(p), denoted by C_k, can be found recursively using $C_k = B_1 C_{k-1} + ... + B_p C_{k-p}$, with C_0 the matrix of contemporaneous responses and where B_j are the VAR coefficients.

wanting to compare these responses of the TM to a SVAR can only estimate a possibly lower-order SVAR(q) in the observed variables alone. Like the TM, the impulse responses at the $j'th$ horizon from the SVAR(q) are a function of its contemporaneous responses and the dynamics of the model. So the quality of the match is influenced by both how the contemporaneous responses differ between the TM and the observables-SVAR, and how the VAR dynamics differ across the models. The latter is generally described as the *VAR truncation error* and is what most of the literature examines. It arises as the observed variables are only a subset of the core variables. Essentially, there is a misspecification due to missing latent variables, although this may be ameliorated by using a higher-order SVAR. The question is how important such misspecification is for a SVAR with a lag length typical of applied macroeconomic work - i.e. how much distortion in the estimated impulse responses does the truncation produce? And are there particular model characteristics associated these distortions?

Section 2 looks at the question of whether the VAR underlying the TM can be well approximated by a VAR in the observed variables using 2 lags. Three examples are chosen from the literature to examine what model characteristics influence the extent of truncation error and to illustrate several points. The first example is of a basic Real Business Cycle (RBC) model with just a technology shock. In this model the core observed variable is taken to be a flow (output) while the latent core variable is a stock (capital). Its simplicity highlights one of the most common issues relating to approximation of the dynamics, namely the omission of stock variables. This issue arises as most SVARs are only formulated with flow variables. In this context it is found that a very low-order VAR provides an excellent approximation to the TM.

What are the consequences of omitting stock variables on the ability to make a match in larger models? The second example in Section 2 uses a small-open economy DSGE model due to Justiniano and Preston (2010). There is one unobserved stock variable - the level of external debt. Unlike the simple RBC model case, some of the impulse responses do differ when using just observable variables. The reasons for this are discussed; of particular note is the fact that a risk premium depending on the external debt position is a common way of closing a small-open economy DSGE model and the omission of the stock of debt in the SVAR diminishes the strength of this mechanism.

Finally, in Section 2 we examine the canonical New-Keynesian model of the US economy, Smets and Wouters (2007). Here the unobserved variables are the two stocks of capital (the actual value and that from a flex-price economy used to construct potential output) and the price of capital. When examining this model Liu and Konstantinos (2012, p.89) found that " ... the truncation bias is the dominant source of the bias in the estimated impulse response functions". We find that such divergences between the impulse responses are lessened a great deal when the monetary policy reaction function is re-specified so that it does not depend on the flex-price gap and its difference, but rather the disequilibria in the actual economy. This alternative specification is common in many estimated DSGE models.

Today TMs often include permanent shocks - typically the log of technology - with transitory disturbances, such as monetary policy shocks. These permanent shocks result in unit roots being present in many of the variables, and so the latter will have a common permanent component. Consequently these TMs involve co-integration between the core endogenous variables and an exogenous latent variable. We discuss the two problems that can arise in the SVAR approximation in this instance.

The first problem reflects the fact that DSGE models with permanent shocks are usually estimated using the first differences of the $I(1)$ variables and the levels of the stationary variables. A researcher, however, estimating a SVAR formulated in these observed variables would encounter a specification error. This specification error arises because the SVAR in observables ignores the fact that the $I(1)$ variables are driven by latent error-correction (EC) processes. An alternative would be to instead estimate a *latent-factor structural Vector Error-Correction Model (VECM)*. The second problem encountered is the presence of unobserved stock variables.

We examine three TMs that exhibit one or both of these problems. The first is the model by An and Schorfheide (2007), where a close match is found between the impulse responses from their TM and a VAR(2). A second choice is the simple RBC model examined in Poskitt and Yao (2017). In their paper it was found that the impulse responses of the observable-variables SVAR departed in a major way from the TM's. Some of this discrepancy is a problem with their graphical display of responses, but one of the responses does show a significant difference. We explain why a difference arises in this instance, and not in the An and Schorfheide (2007) model.

The final paper with permanent shocks considered is Erceg, Guerrieri and Gust (2005). They found that a finite-order VAR in the observable variables could recover impulse responses accurately when the TM used had an RBC orientation, but not when sticky prices were present. We argue that, while the introduction of sticky prices increases the order of the VAR, this is not the source of the approximation problems. Rather, these arise from the simultaneous introduction of variable capital utilisation into the model. Variable capacity utilisation changes the nature of capital services since it involves both a utilization rate and a capital stock. Only one of these variables can be substituted out of the model, leaving one unobserved, which increases the truncation error. An implication of this is that data on capital utilization may be an important variable to be included in a VAR if it is a property of the TM.

In general we find that the truncation issue may not be that important if the variables of the VAR are carefully chosen and it might also be ameliorated by using latent-variable VARs.

In the analysis of section 2 and 3 it is assumed that the contemporaneous impulse responses from the SVAR are identical to those from the TM. This was done so as to focus solely on the extent of truncation error, and is a strategy used by Kapetanios et al. (2007) and Ravenna (2007). However, differences between the contemporaneous responses of the TM and SVAR would also influence the ability to make a match. Put another way, it is possible to get differing impulse responses from the SVAR and the TM even though the VAR approximates the dynamics of the TM quite well. The specification of the contemporaneous impulse responses is essentially an issue arising from the *definition of structural shocks*. Section 4 therefore considers this final dimension for making a match, and the difficulties in doing so with just the observable data. Fundamentally, the difficulty is that researchers with a SVAR in the observable variables often seek to have a great deal of flexibility in dynamics, and this can create problems when trying to estimate the contemporaneous responses consistent with the TM because the identification restrictions embodied in some TMs are not often used in the SVAR literature. Finally, Section 5 concludes.

In summary, the focus of this paper is to assess the ability of a SVAR in the observable variables to make a match with the impulse responses from a TM. A variety of TMs are considered from the literature with differing characteristics. In general it is found that a reasonable match may be obtained, with divergences often primarily occurring not due to differing dynamics between the models but due to the identifying restrictions required to measure the shocks.

2 What are the Issues in Approximating a TM with an Observable-Variables SVAR? The I(0) Case

2.1 Analysis

This section provides some simple analysis to illustrate the issues that can arise when approximating a TM with a SVAR in the observed variables alone.

Historically all variables in DSGE models were taken to be $I(0)$, so we start with an analysis of the VAR implied by the TM in this case. Let z_t be the core endogenous and exogenous variables in a TM. In most instances the TM has the structural equations[4]

$$A_0 z_t = CE_t(z_{t+1}) + A_1 z_{t-1} + u_t, \tag{1}$$

where u_t are shocks possibly following a VAR(1), $u_t = \Phi u_{t-1} + \varepsilon_t$, and ε_t is a vector of white noise processes with covariance matrix Ω that is diagonal. This system can then be solved for z_t by using (for example) the method of undetermined coefficients. This produces a solution[5]

$$z_t = Dz_{t-1} + Gu_t. \tag{2}$$

Equation (2) is what we will call an incomplete VAR since it requires a specification for the exogenous shocks u_t. Mostly these are taken to be autoregressive (AR) processes. Consequently, using $u_t = \Phi u_{t-1} + \varepsilon_t$ we get the complete VAR

$$\begin{aligned} z_t &= Dz_{t-1} + G\Phi u_{t-1} + G\varepsilon_t \\ &= Dz_{t-1} + G\Phi G^+(z_{t-1} - Dz_{t-2}) + G\varepsilon_t \\ &= (D + G\Phi G^+)z_{t-1} - G\Phi G^+ Dz_{t-2} + G\varepsilon_t \\ &= B_1 z_{t-1} + B_2 z_{t-2} + C_0 \varepsilon_t, \end{aligned} \tag{3}$$

where $G^+ = (G'G)^{-1}G'$ if there are not more shocks than variables.[6]

---

[4]There are very few TMs that cannot be written in this way. If the model equations have more than one lag in variables then we would need to expand z_t in Equation (1) to contain lagged variables. The analysis would still proceed in the same way but it would be necessary to select the current values of variables from the augmented z_t vector.

[5]See Binder and Pesaran (1995). The conditions for the solution are twofold: a rank condition and the Blanchard-Kahn stability conditions must be satisfied (Blanchard and Kahn 1980). Users of Dynare will be familiar with the program checking these conditions.

[6]If the shocks u_t follow a moving average process (as in Smets and Wouters 2007) then the observable variables will follow a Vector Autoregression Moving-Average (VARMA) process.

This latter equation will be referred to as a *semi-structural* VAR(2) (SSVAR(2)) in the sense that the dynamics are described by a VAR while the VAR disturbances are $e_t = G\varepsilon_t = C_0\varepsilon_t$. The contemporaneous response of z_t to the structural innovations ε_t, is C_0,. In sections 2 and 3 we are going to set C_0 for the SSVAR in observables to that from the TM. It is clear that once C_0 is given then the SSVAR is the same as a VAR and an SVAR, and so we will sometimes use these terms interchangeably. It is often convenient to work with the incomplete VAR in Equation (2); at other times the complete VAR in Equation (3) is preferable. .

The problem in practice is that not all of the variables in the TM, z_t, are observable and SVAR models are traditionally specified in terms of observed variables alone. Hence, we partition z_t into those that are observed, z_t^o, and unobserved, z_t^u. In this case the incomplete VAR, Equation (2) can be written as[7]

$$z_t^o = D_{oo}z_{t-1}^o + D_{ou}z_{t-1}^u + G_o u_t \quad (4)$$
$$z_t^u = D_{uo}z_{t-1}^o + D_{uu}z_{t-1}^u + G_u u_t. \quad (5)$$

In what follows we will assume that $u_t = \varepsilon_t$ i.e. $\Phi = 0$. If this was not the case then we would just add z_{t-2}^o and z_{t-2}^u to the equations but the method now described is unchanged.

In most TMs the number of shocks equals the number of observed variables. In that case

$$\varepsilon_t = G_o^{-1}(z_t^o - D_{oo}z_{t-1}^o - D_{ou}z_{t-1}^u),$$

and so

$$\begin{aligned} z_t^u &= D_{uo}z_{t-1}^o + D_{uu}z_{t-1}^u + G_u G_o^{-1}(z_t^o - D_{oo}z_{t-1}^o - D_{ou}z_{t-1}^u) && (6)\\ &= F_1 z_t^o + F_2 z_{t-1}^o + F_3 z_{t-1}^u && (7)\\ &= \sum_{j=0}^{\infty} H_j z_{t-j}^o, && (8)\end{aligned}$$

where $F_1 = G_u G_o^{-1}, F_2 = (D_{uo} - G_u G_o^{-1} D_{oo})$ and $F_3 = (D_{\text{uu}} - G_u G_o^{-1} D_{ou})$. This demonstrates it is possible to recover the unobserved variables from the observed variables using their contemporaneous values and enough lags.

How many lags are enough to approximate the TM in terms of the observable variable alone? This depends upon the magnitude of the H_j- which depends on the eigenvalues of F_3. In the case where the latent variables are z_t^u stocks there is the potential that one might need a large number of lags of the observed variables in order to capture the unobserved.

Substituting Equation (8) into Equation (4) one gets

$$z_t^o = D_{oo}z_{t-1}^o + D_{ou}(\sum_{j=0}^{\infty} H_j z_{t-j}^o) + G_o\varepsilon_t$$

If we choose only a VAR(2) in observables then the term $\psi_t = D_{ou}\sum_{j=3}^{\infty} H_j z_{t-j}^o$ will be excluded from the regression used to fit it Consequently, when $\{z_{t-k}\}_{k=1}^2$ and ψ_t are uncorrelated then there will be no bias in the estimated coefficients of $\{z_{t-k}^o\}_{k=1}^2$. Nevertheless, the

[7] Note that we have not solved for the complete VAR as the error term in this form is u_t and not ε_t. But, as shown previously in Equation (3), converting to a VAR that has ε_t only raises the order of the VAR by one.

impulse responses to the shocks ε_t would be incorrect because of the omission of the higher-order lags in z_t^o from the system when impulse responses are computed. When the variables are correlated there is also a bias in the estimates of the coefficients of $\{z_{t-k}\}_{k=1}^2$. Hence the misspecification has two effects which will depend upon the magnitude of H_j $(j = 3, ...)$ and the correlation of $\{z_{t-k}\}_{k=1}^2$ with ψ_t.

A problem is that the magnitude of both of these effects will be model dependent. Our strategy is to examine a range of models from the literature so as to gain an indication of the cases where they can affect the possibility of "making a match" of the impulse responses. Indeed, we will see that, even where it seems that the unobserved variables can be reconstructed from the observed variables - since the R^2 from regressing z_t^u on z_t^o, z_{t-1}^o is greater than .95 - the term left out of the approximation $\sum_{j=3}^{\infty} H_3 z_{t-j}^o$ can lead to a large bias in the estimated dynamics of the VAR(2) in z_t^o.[8] It is not enough to have an R^2 from this regression close to unity.

2.2 Illustration: A Simple Real Business Cycle Model

To demonstrate the results mentioned above suppose we take the basic RBC model in Uhlig (1999) as the TM. This has the equations (where investment has been substituted out)

$$
\begin{aligned}
l_t &= y_t - c_t && (9)\\
\frac{C^*}{Y^*}c_t + \frac{K^*}{Y^*}k_t &= y_t + (1-\delta)\frac{K^*}{Y^*}k_{t-1} && (10)\\
c_t &= E_t(c_{t+1} + r_{t+1}) && (11)\\
R^* r_t &= \alpha \frac{Y^*}{K^*}(y_t - k_{t-1}) && (12)\\
y_t &= (1-\alpha)a_t + \alpha k_{t-1} + (1-\alpha)l_t && (13)\\
a_t &= \rho_a a_{t-1} + \varepsilon_t^a. && (14)
\end{aligned}
$$

Here small letters represent log deviations from steady state, * are steady-state values, c_t is consumption, k_t is the capital stock, r_t is the gross real rate of return, a_t is an AR(1) technology shock with parameter ρ_a, and l_t is labour input. The parameters are set to $\alpha = .4, \delta = .025, \rho_a = .9, R^* = .99$, and the steady-state values are functions of these parameters. The core observed endogenous variable will be taken to be output and the latent variable that is the capital stock. Given these c_t and l_t can be substituted out. The

[8] A popular approach has been to write the system in state-space form, separating the state and observed variables - the ABCD representation - before finding the conditions for a finite-order VAR representation (Fernández-Villaverde, Rubio-Ramírez, Sargent and Watson 2007). This approach was adopted by Ravenna (2007), Franchi and Vidotto (2013) and Morris (2016). Giacomini (2013) gives a survey of the literature emphasizing the ABCD representation and the conditions for a finite-order representation. The focus of our analysis is not about finding such conditions. Rather our focus is to point out that in many models a high-order VAR may not be needed to capture impulse responses from a TM, at least for the horizons which are of the greatest interest to policy makers. Furthermore, we examine which characteristics of the TM influence the quality of the match. One difference between the ABCD representation and ours is that the shocks in Equations (4) and (5) are left as u_t rather than the innovations ε_t as in the ABCD approach, since there is no need to describe the nature of u_t when considering the impact of omitted variables.

exogenous variable is a_t. The equivalent of Equations (4) and (5) for the RBC model are

$$y_t = .133k_{t-1} + 1.21a_t \quad (15)$$
$$k_t = .95k_{t-1} + .094a_t. \quad (16)$$

The true impulse response function of y_t to ε_t^a from the RBC model solution and that from an AR(2) in y_t are very close, leading to the question of why it is not necessary to know the latent capital stock in this case.[9] The value for F_3 for the solution to the RBC model suggests that extraction of a good measure of the capital stock from output data would require many lags of that variable; indeed regressing k_t against y_t, y_{t-1} and y_{t-2} gives an $R^2 = .56$, which increases to .95 when there are 20 lags of y_t. This indicates that to reconstruct the latent capital stock with observed output alone requires many lags. However, once the unobservable capital stock has been written as a function of the observable y_t, the AR equation for y_t is

$$\begin{aligned} y_t &= D_{ou}(I - F_3L)^{-1}F_1y_{t-1} + 1.21a_t \\ &= .133 \times .0777 \times \sum_{j=0}^{\infty}(.94)^j y_{t-1-j} + 1.21a_t \\ &= .01y_{t-1} + .0097y_{t-2} + .00913y_{t-3} + .0086_{t-4} + ... + 1.21\varepsilon_t^a, \end{aligned}$$

It is evident that any bias in dynamics from omitting $.00913y_{t-3} + .0086_{t-4} + ...$ from the AR(2) regression will be small. In fact, the regression of $.00913y_{t-3} + .0086y_{t-4} + ...$ on y_{t-1} gives a coefficient of .023. Even an AR(1) in y_t will capture the RBC model impulse response very well. Thus, even very high values of F_3 do not preclude a low-order VAR from providing a good approximation to the impulse responses from the TM. This is a reflection of the small values of F_1 and D_{ou} for the RBC model.

There are many variants to the basic RBC model above that have appeared in the literature. One of these adds a second shock; for example, a preference shock. As there are now two shocks, an additional observed variable can be added (so as to make the number of shocks and observed variables equal). We will now assume that the observed variables are y_t and l_t. Then the capital stock at the beginning of t, k_{t-1}, can be found through the production function by combining these core variables and the exogenous level of technology a_t Hence by careful choice of the observed variables the capital stock no longer appears as an unobserved variable and can be eliminated from the core variables. In this case a VAR(1) in the core observable variables y_t and l_t will exactly reproduce the impulse responses of the TM. This will be seen in later examples.

2.3 Illustration: The Justiniano-Preston (2010) Model

The important role of stock variables in influencing the ability to make a match between the impulse responses is also evident in small-open economy models. Justiniano and Preston (2010) present a small-open economy DSGE model that has thirty-four endogenous variables and twelve shocks. Twelve variables were taken to be observable - domestic output, an

[9] Equating Equations (15)-(16) with (4)-(5) we have $D_{uu} = .95, D_{ou} = .133, D_{uo} = D_{oo} = 0, G_u = .094$, and $G_o = 1.21$, leading to $F_1 = .0777, F_2 = 0$, and $F_3 = .94$.

interest rate and inflation (y_t, i_t and π_t respectively), and their foreign counterparts, - denoted by an $*$ - the nominal and real exchange rates (s_t and q_t), real wages (w_t and w_t^*) and hours (h_t and h_t^*). Its solution is a SSVAR(2). An example of one of the equations coming from using the Justiniano-Preston (2010) model as the TM is that for output, namely

$$\begin{aligned} y_t &= 1.49y_{t-1} - .52y_{t-2} - .026y_{t-1}^* + .017y_{t-2}^* - .032i_{t-1}^* + .036\pi_{t-1}^* \\ &\quad -.0004\pi_{t-2}^* + .004w_{t-1}^* - .001w_{t-2}^* + .003h_{t-1}^* \\ &\quad +.241i_{t-1} - .06q_{t-1} + .06q_{t-2} - .22\pi_{t-1} + .004\pi_{t-2} - .04w_{t-1} \\ &\quad +.018w_{t-2} - .024h_{t-1} - .04s_{t-1} + .036s_{t-2} - .0001B_{t-1} + e_{yt}, \end{aligned} \tag{17}$$

where the VAR error e_{yt}, is a function of the structural shocks

$$\begin{aligned} e_{yt} &= -.034\varepsilon_t^a - .011\varepsilon_t^{a^*} + .17\varepsilon_t^g + .042\varepsilon_t^{g^*} - 1.495\varepsilon_t^i + .094\varepsilon_t^{i^*} \\ &\quad +.09\varepsilon_t^{cp^*} - .386\varepsilon_t^{cph} - .022\varepsilon_t^{cpf} + .0002\varepsilon_t^n + .00005\varepsilon_t^{n^*} + .65\varepsilon_t^{rp}. \end{aligned} \tag{18}$$

In Equations (17) and (18) the structural shocks are for preferences ($\varepsilon_t^g, \varepsilon_t^{g^*}$), technology in both economies ($\varepsilon_t^{a^*}, \varepsilon_t^a$), monetary policy ($\varepsilon_t^i$ and $\varepsilon_t^{i^*}$), cost push shocks in the foreign economy and to the foreign and domestic goods in the domestic economy ($\varepsilon_t^{cp^*}$, $\varepsilon_t^{cph}, \varepsilon_t^{cpf}$ respectively), labour supply (ε_t^n and $\varepsilon_t^{n^*}$) and the risk premium (ε_t^{rp}). Even though there are thirty-four endogenous variables in the DSGE model twenty-one can be substituted out through identities, leaving just thirteen for the core set. This results in a single unobserved variable, the level of net foreign assets (B_t). This is the only stock variable in the TM as capital is not included in the model.

Equation (17) is an *identity* and is a SSVAR. Hence, even though it is not a structural equation, it contains information about the impact of structural shocks. Specifically, the contemporaneous impulse response matrix C_0 can be computed from the coefficients attached to ε_t in Equation (18). Thus the contemporaneous impact of a monetary shock on output is -1.495.[10]

Figure 1 presents the impact of a domestic monetary shock upon inflation using the SSVAR(2) with all variables from the TM as well as one with just the observable variables. There is very little difference in the responses. Figure 2 shows the response of the real exchange rate to the monetary shock and, again, the two responses are similar. The responses of the real exchange rate and output to risk premium shocks (not shown) are also very close.

In contrast, Figure 3 shows the response of output to the domestic technology shocks; in this case there is a difference, especially as the horizon lengthens. In particular, it takes far longer for the observable-variables SSVAR estimates of the responses to die away. This pattern can be seen in a number of other impulse responses, such as the response of the real exchange rate to technology shocks. Essentially, in relation to real shocks, such as technology, the complete system returns to the steady-state position much faster with the "all variables" SSVAR than for an SSVAR which omits foreign asset balances.

The logic for this failure to make a match is that the accumulation of debt is important for ensuring that the real exchange rate so that debt converges to its steady state. The

[10] In all the work of this paper data was simulated from the TM and then the identity governing the evolution of variables was found by fitting a regression.

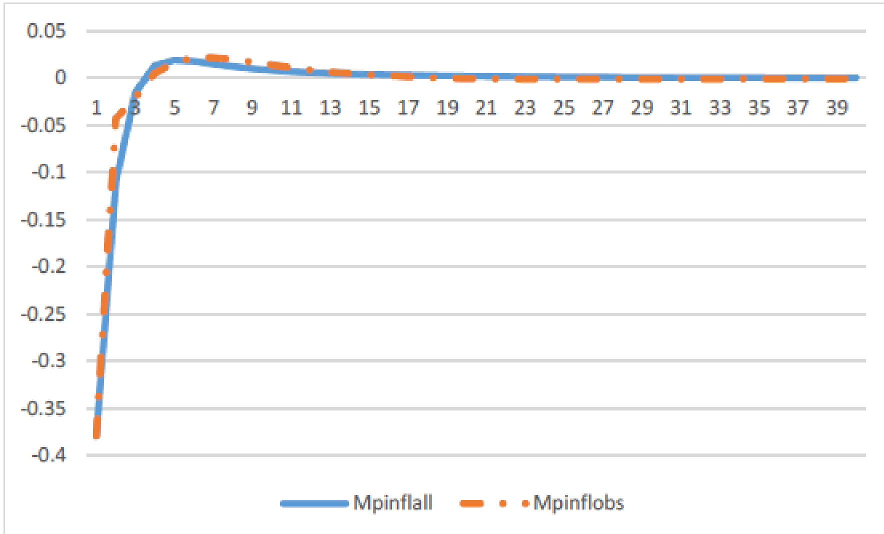

Figure 1: Response of Inflation to a Monetary Shock Response of Inflation to a Monetary Shock for SVAR(2)s Fitted to Justiniano-Preston Model Output with All and with just Observed Variables

inclusion of a debt-elastic premium is a common way of closing small-open DSGE models; see Schmitt-Grohe and Uribe (2003) for a discussion of the methods of inducing such a stabilizing mechanism. Without such a mechanism - and it will be absent in an SVAR that does not include debt - we would expect that convergence to any steady state will be far slower, and might not even occur. Indeed, it is instructive to note that the estimated dynamics of the VAR equation for the real exchange rate change markedly when debt is included in the system to when it is omitted. This suggests that debt is highly correlated with the included variables and this is the case - regressing debt on the observable variables produces an R^2 of .99. Consequently, omitting debt from the system will produce biased estimates of the dynamics.

This result sheds light on the findings of Kapetanios et al. (2007), who studied a small-open model of the type used by Justiniano and Preston (2010) as their TM. Kapetanios et al. concluded that one needed a VAR of order 50 and 30,000 observations to capture the responses. They also found that it took much longer to return to the steady state. The TM contained 26 variables, but only 6 were taken to be observed. Foreign assets were not observed, and our results suggest that will have been an important factor contributing to the difficulties they encountered in making a match.

In all, it appears that the latent level of debt in SVARs of small-open economies is very likely to be an issue in getting a correct measure of the dynamics of real shocks at longer horizon with the SSVAR in the observable variables, and so every effort should be made to include such a variable into small-open economy SVARs. If it is difficult to measure the debt variable one possibility might be to include both absorption and domestic output as observed variables in the VAR, and to then treat debt as a latent variable that reacts to these observable variables. The difference between absorption and domestic output captures

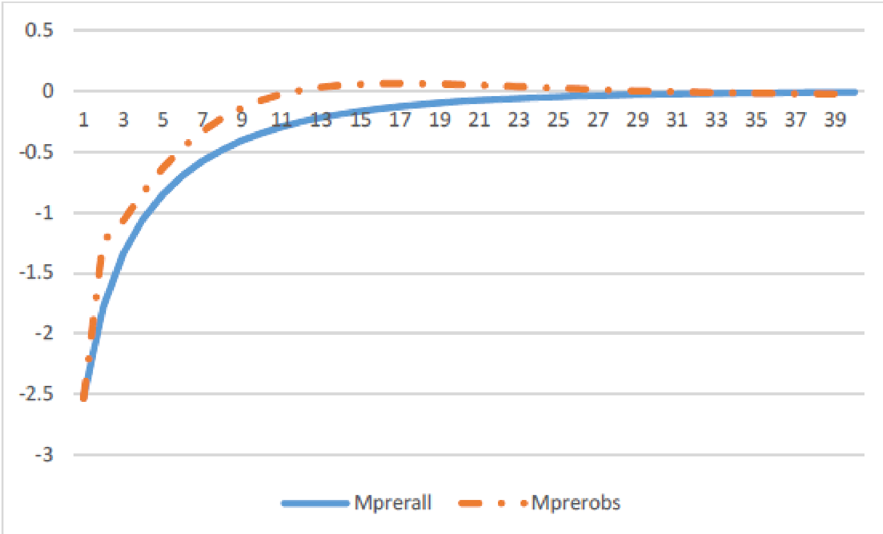

Figure 2: Response of the Real Exchange Rate to a Monetary Shock for SVAR(2)s Fitted to Justiniano-Preston Model Output with All and with just Observed Variables

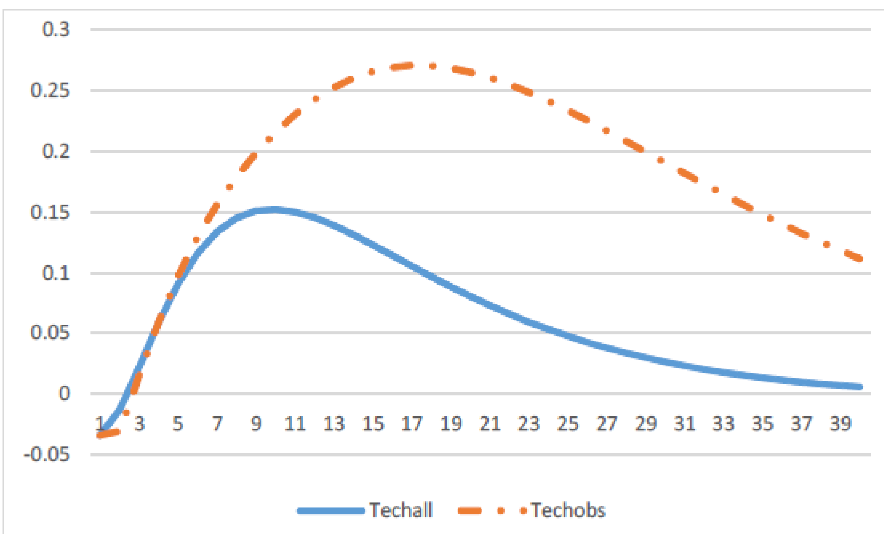

Figure 3: Response of Output to a Domestic Technology Shock for SVAR(2)s Fitted to Justiniano-Preston Model Output with All and with just Observed Variables

the current account balance and hence the evolution of the stock of debt.

2.4 Illustration: The Smets and Wouters (2007) Model

Smets and Wouters (2007) (SW) is a New-Keynesian model of the United States economy which has been extensively used in the literature, and adopting it as the TM provides extra insights into the relationships between the different modelling approaches. It has a large number of endogenous variables (twenty-four) but only seven shocks and observable variables. In its original form the mark-up shocks in the Phillips curves are ARMA processes so, by definition, the solution will not be a finite-order VAR. However, in this section we replace those ARMA terms with standard AR processes. Fourteen endogenous variables can be substituted out using identities, implying that the SW model can be reduced to a SSVAR(2) in ten core endogenous variables.

In the SW model seven of its ten core variables are directly observed, namely the logs of output y_t, consumption c_t, investment i_t, hours h_t, inflation π_t, wages w_t and the interest rate r_t. This leaves only three unobserved endogenous variables - capital k_t, the flex-price level of capital kf_t, and the price of capital pk_t. The production function in Smets and Wouters has capital services in it, which can be found from output, hours and technology. However, these are allowed to vary with utilization of the the lagged capital stock. Consequently, either the capital stock (or the utilization rate) are not observable and will be a latent core variable.

We expect F_3 in Equation (7) to have at least two large eigenvalues, reflecting the unobserved capital stocks. To assess the ability to proxy these latent variables with the observed data we regress k_t, kf_t and pk_t against two lags of the observable variables and obtain R^2 values of .96, .53 and .08 respectively, indicating that it varies considerably. Offsetting this, however, is the fact noted previously that a low R^2 would mean less of a bias in the parameter estimates when the latent variables are omitted in the estimation of the VAR.

The effects of monetary shocks upon inflation are shown in Figure 4 and, just as for the Justiniano-Preston model, these are little affected by the use of a VAR in the observable variables alone. In a similar vein, real shocks do have different response functions, as seen in Figure 5 for output and the technology shock, but by far less than was apparent in the Justiniano-Preston model. It is notable that the observable-variables VAR now has impulse responses which converge to zero much faster than the TM. The reason for this is that much of the persistence comes from the omitted stock variables. Unlike the case of foreign debt, capital stocks do not act as a stabilizing device but rather make the adjustment longer. The impulse responses for the level of output to what Smets and Wouter term an "exogenous spending shock" are shown in Figure 6 and, as noted in Liu and Konstantinos (2012), there is a relatively large discrepancy between those from the TM and a SVAR(2) in observable variables.

To explore these results further we note that the sole place the flex-price variables enter the sticky-price economy is through the Taylor rule, which in SW depends on a flex-price measure of the output gap, $(y_t - yf_t)$. This is the only reason for kf_t, which is latent, appearing in the solution to the model. While theoretically founded, in many applied models the interest rate is instead related directly to y_t, the log deviation of output from its steady state. Hence we convert the SW model to one with such a Taylor rule. Figures 7 and 8 show the same impulse responses as in Figures 5 and 6 omitting the flex-price terms and therefore

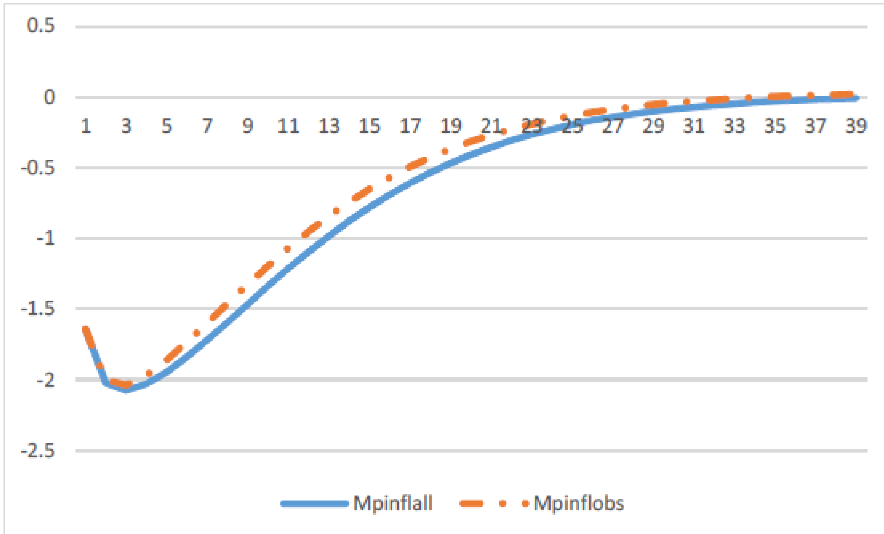

Figure 4: Response of Inflation to a Monetary Shock for SVAR(2)s Fitted to Smets-Wouters Model Output with All and with just Observed Variables

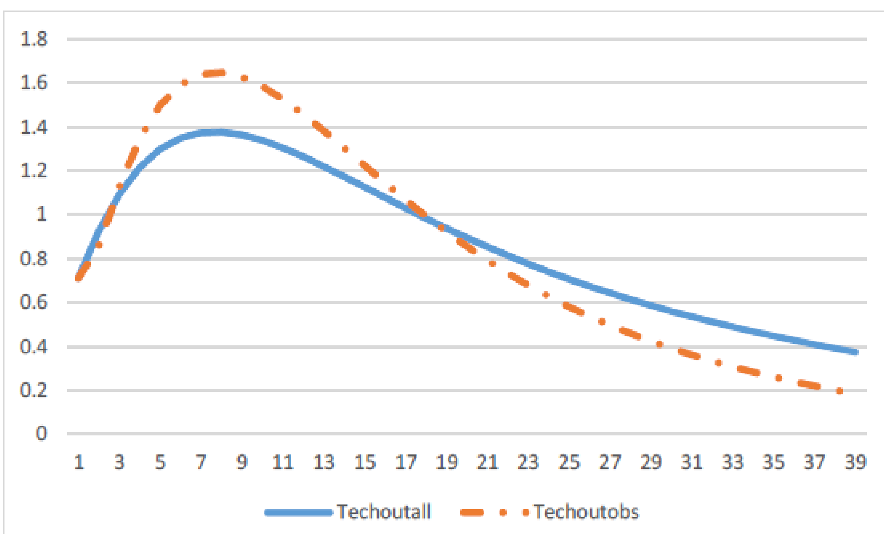

Figure 5: Response of Output to a Technology Shock for SVAR(2)s Fitted to Smets-Wouters Model Output with All and with just Observed Variables

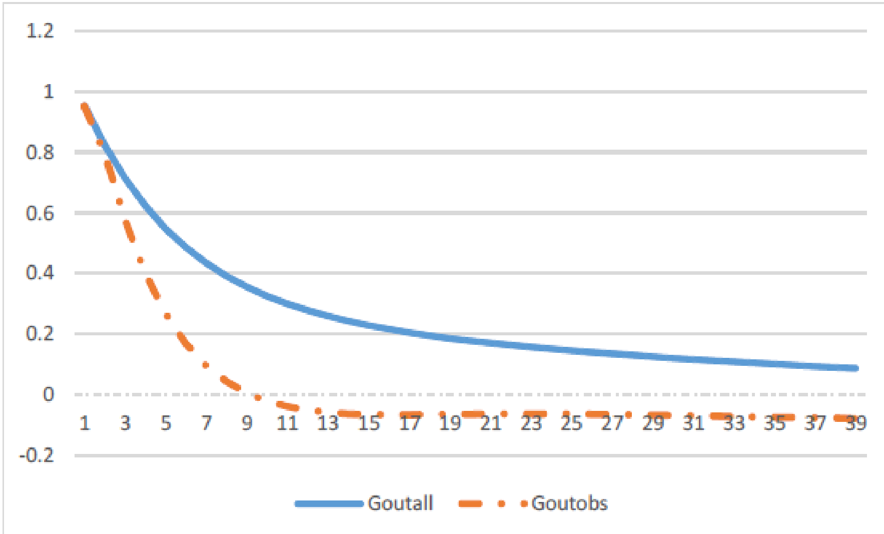

Figure 6: Response of Output to a an Exogenous Spending Shock for SVAR(2)s Fitted to Smets-Wouters Model Output with All and with just Observed Variables

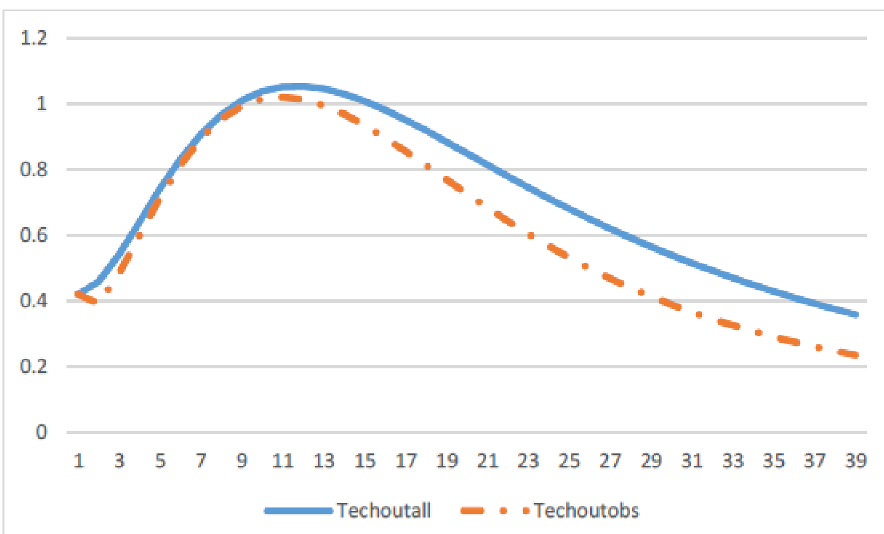

Figure 7: Response of Output to a Technology Shock for SVAR(2)s Fitted to Smets-Wouters Model (With no Flex-Price) Output with All and with just Observed Variables

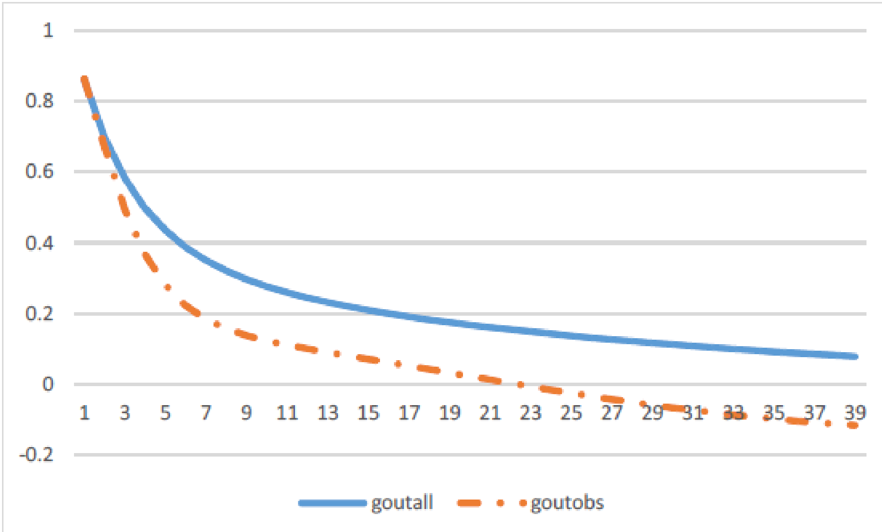

Figure 8: Response of Output to a an Exogenous Spending Shock for SVAR(2)s Fitted to Smets-Wouters Model (With no Flex-Price) Output with All and with just Observed Variables

decreasing the number of latent variables. The observable-variables SSVAR(2) responses are now much closer to those for the TM. Interestingly, monetary policy shocks have similar impulse responses for either variant of the TM. So the inclusion of the flex-price output gap in the Taylor rule in SW plays an important role in limiting the ability of a VAR to make a match.

3 Truncation Bias when the TM and Observable Variables SVAR Contain Cointegration

3.1 Analysis

Traditionally the decision rules in TMs have been log-linearized prior to estimation. This approximation is done about the steady-state, so effectively the variables are expressed relative to their steady-state value. An example is a consumption Euler equation with log utility, namely $C_t^{-1} = \beta E_t C_{t+1}^{-1} R_{t+1}$, where C_t is the level of consumption, β the discount factor and R_t is a real interest rate. Written in terms of variables normalised relative to their steady-state, namely C^* and R^*, the Euler equation is $(\frac{C_t}{C^*})^{-1} = \beta R^* E_t(\frac{C_{t+1}}{C^*})^{-1}\frac{R_{t+1}}{R^*}$.

It has become increasingly common for TMs to include permanent shocks. An example is if the log-level of technology, a_t, is assumed to follow a unit root. In this case a different normalisation, rather than a fixed steady-state level, has to be used to reflect that the economy has a long-run growth path. In this example the divisor would be $A_t \equiv \exp(a_t)$, and the normalised consumption Euler equation would be $(\frac{C_t}{A_t})^{-1} = \beta R^* E_t(\frac{C_{t+1}}{A_{t+1}})^{-1}(\frac{A_{t+1}}{A_t})^{-1}\frac{R_{t+1}}{R^*}$.

Then, after log-linearization, it becomes

$$c_t - a_t = E_t[c_{t+1} - a_{t+1} + \Delta a_{t+1}] - E_t r_{t+1} + \ln r^*,$$

where the lower case letters represent the logs of the upper case ones. Furthermore, many modern TMs additionally allow for persistence in technology growth, i.e.

$$\Delta a_t = \rho_a \Delta a_{t-1} + \varepsilon_{at}.$$

This is particularly the case for models used in a policy environment.

Under such a specification $E_t \Delta a_{t+1} = \rho_a \Delta a_t$, making the linearized consumption Euler equation:

$$c_t - a_t = E_t[c_{t+1} - a_{t+1}] + \rho_a \Delta a_t - E_t r_{t+1} + \ln r^*.$$

It is apparent that the inclusion of a permanent technology shock results in some variables in these models being I(1) and co-integrated. An example of co-integration is between c_t and a_t since $c_t - a_t$ is I(0). All the model variables which are I(1) are expressed as deviations from a_t, a process often referred to as "stationizing". Thus, in terms of the RBC model of the previous section, if a_t was a unit root process, we would have variables $y_t - a_t, c_t - a_t$ etc.. These variables are then $I(0)$ and represent the error-correction (EC) terms. The consequence is that there is co-integration between the *three* variables y_t, c_t, and a_t. These EC terms would be present among the core model variables, z_t, and, after the TM is solved, there will be a VAR in z_t.

The VAR in z_t will contain observed and unobserved variables as before. However, in this case the "stationized" variables $y_t - a_t$ etc. (denoted with a superscript S), while not directly observed, may be related to the observed data. As an example, let Δc_t be observed data on consumption growth. Then

$$\Delta c_t = \Delta(c_t - a_t) + \Delta a_t = \Delta c_t^S + \Delta a_t.$$

In contrast, there can be some other stationized variables that do not relate directly to data and which require a model for their construction, such as the capital stock when there is a variable utilization rate. With the first type of variable it is necessary to add a statistical specification for the latent exogenous process a_t, but that *does not require an economic model.* Alternatively, to re-construct the capital stock when there is a variable utilization rate we do need such a model.

3.2 Illustration: The An and Schorfheide (2007) Model

To fix these ideas, we consider a model with a permanent technology shock that has become a workhorse for such analysis. Specifically we take the An and Schorfheide (2007) model which was analyzed in Giacomini (2013) and Morris (2017). Here y_t is the log of output, c_t the log of consumption, π_t inflation, r_t the interest rate and g_t a fiscal variable. There are three shocks in the model, namely technology, $\varepsilon_{a,t}$, fiscal, $\varepsilon_{g,t}$, and monetary policy, $\varepsilon_{r,t}$. The

TM has the equations:

$$\begin{aligned}
y_t^S &= E_t(y_{t+1}^S) + g_t - E_t g_{t+1} - \frac{1}{\tau}(r_t - E_t\pi_{t+1} - E_t\Delta a_{t+1}) \\
\pi_t &= \beta E_t(\pi_{t+1}) + \frac{\tau(1-\nu)}{\varpi\pi^2\phi}(y_t^S - g_t) \\
c_t^S &= y_t^S - g_t \\
r_t &= \rho_r r_{t-1} + (1-\rho_r)\psi_1\pi_t + (1-\rho_\tau)\psi_2(y_t^S - g_t) + \rho_r\varepsilon_{r,t} \\
\Delta a_t &= \rho_a\Delta a_{t-1} + \sigma_a\varepsilon_{a,t} \\
g_t &= \rho_g g_{t-1} + \sigma_g\varepsilon_{g,t} \\
y_t^S &= y_t - a_t.
\end{aligned}$$

We begin by examining the VAR representation of the solution of the TM in terms of the stationized variables. Assuming that y_t^S, π_t and r_t are the core endogenous variables in the TM, the system can be simplified by recognizing that c_t^S is a function of y_t^S and the exogenous variable g_t, so it can be substituted out. The resulting solution is a VAR(1) in y_t^S, π_t and r_t.

The observable variables used for estimation, however, *are not* the stationized variables, y_t^S, π_t and r_t but $\Delta y_t, \pi_t$ and r_t. It is clear that the TM implies that there will be co-integration between y_t, c_t and a_t.

Adopting the parameter values from Giacomini (2013), the solution for the TM is the SSVAR(1)

$$\begin{aligned}
y_t^S &= .95y_{t-1}^S - .5\pi_{t-1} + .1945r_{t-1} + .0037\varepsilon_t^a + .006\varepsilon_t^g - .0019\varepsilon_t^r \\
\pi_t &= .616\pi_{t-1} - .114r_{t-1} + .0037\varepsilon_t^a - .0012\varepsilon_t^r \\
r_t &= .776r_{t-1} + .308\pi_{t-1} + .0018\varepsilon_t^a + .0013\varepsilon_t^r.
\end{aligned}$$

However this involves the latent variable y_t^S and so it is *not* a VAR in the observable variables alone. What is observed instead is Δy_t, so we need to ask whether $\Delta y_t, \pi_t$ and r_t follow a SSVAR?

Using the measurement equation

$$\Delta y_t = \Delta y_t^S + \Delta a_t, \tag{19}$$

and the relation between π_t, a_t and r_t of $\pi_t = 1.79\Delta a_t - .92r_t$, the solution for Δy_t can be expressed as

$$\Delta y_t = .0018\pi_{t-1} + .6568r_{t-1} - .05y_{t-1}^S + .0067\varepsilon_t^a + .006\varepsilon_t^g - .0019\varepsilon_t^r. \tag{20}$$

That is, it is an equation from a VECM that has y_{t-1}^S as the lagged error-correction term. Consequently, if the system is expressed as a SSVAR in just the observable variables, the term $-.05y_{t-1}^S$ would be ignored and this would be a specification error.

This analysis demonstrates the fact that a SSVAR in observable variables will rarely be the appropriate way to proceed if data is generated by a TM with non-stationary technology. For example, if one thought of the SW model as having a unit root in technology, then there would be stationized variables such as $(y_t - a_t), (c_t - a_t), (i_t - a_t)$ etc..[11] These could be transformed to $(c_t - y_t), (i_t - y_t)$ and $(y_t - a_t)$, but there will always be one unobservable EC term that would be missing from an observable-variable VECM.[12]

How important is the specification error of ignoring the EC term represented by y^S_{t-1}? In this case it will depend on the relative variances of $-.05y^S_{t-1}$ and $.0067\varepsilon^a_t + .006\varepsilon^g_t - .0019\varepsilon^r_t$. The latter is some ten times the former and so we might expect that there will only be a small difference between the impulse responses from the observable-variables SSVAR and those from the TM. If one fits a MA(1) to the equation that has Δy_t as dependent variable and $\Delta y_{t-1}, r_{t-1}$ and π_{t-1} as independent variables the MA coefficient is -.06, showing that there is only a small degree of serial correlation in the residuals of the SSVAR(1) equation for Δy_t. Consequently, if a SSVAR(2) is fitted to the observable variables this could be expected to capture quite accurately what MA error there is in the equation. Figure 9 demonstrates this is the case. It shows the cumulative impact of a technology shock upon the level of output for both the TM and $SSVAR^{obs}(2)$ models. It is only when the horizon is very long that one sees small differences.[13]

The underlying logic is that a small coefficient on the error-correction term means that one can omit that regressor from the Δy_t equation with little effect and so the SSVAR in observable variables alone will be close to the correct representation.[14] Accordingly, a reasonable rule of thumb is that when the EC term coefficient is small we can work with a VAR in differences for the I(1) variables and get a good approximation to the impulse responses.

Essentially we are dealing with a SSVECM model which has $y_t - a_t$ as an error-correction term. Its omission in the observable-variables SSVAR is a misspecification, which might be avoided by fitting a finite-order latent factor VECM which includes y^S_{t-1} in the Δy_t equation. In order to estimate such a system all that is required is a *statistical assumption* about the exogenous variable Δa_t, akin to that in the TM.

3.3 Illustration: The Poskitt and Yao (2017) Model

A RBC model with technology following an I(1) process is taken as the TM by Poskitt and Yao (2017).[15] They have two core observed variables - output, y_t, and hours, h_t- as well as a latent variable, the capital stock. The capital stock, however, can be substituted out using the production function, the observed variables and the exogenous technology process, a_t. Consequently, the only core endogenous latent variable is stationized output, y^S_t.

The TM is a SSVAR(1) in y^S_t and h_t, as was found for the An and Schorfheide (2007) model. When expressed in terms of the observable variables, Δy_t and h_t, a latent variable SSVECM(1) is obtained with the form

$$\Delta y_t = -.01h_{t-1} - .18y^S_{t-1} - 1.56\varepsilon^h_t + .96\varepsilon^a_t \quad (21)$$

$$h_t = .93h_{t-1} - .24y^S_{t-1} - 2.4\varepsilon^h_t + .48\varepsilon^a_t. \quad (22)$$

[11] Such a model was considered by Del Negro, Schorfheide, Smets and Wouters (2007).

[12] The presence of an unobserved EC term was also discussed in Liu, Pagan and Robinson (2018) in the context of the Rees, Hall and Smith (2016) model.

[13] As in the previous models monetary policy impacts on inflation are close for both models.

[14] The EC term does not appear in the other two equations.

[15] Ravenna (2007) studies the same TM.

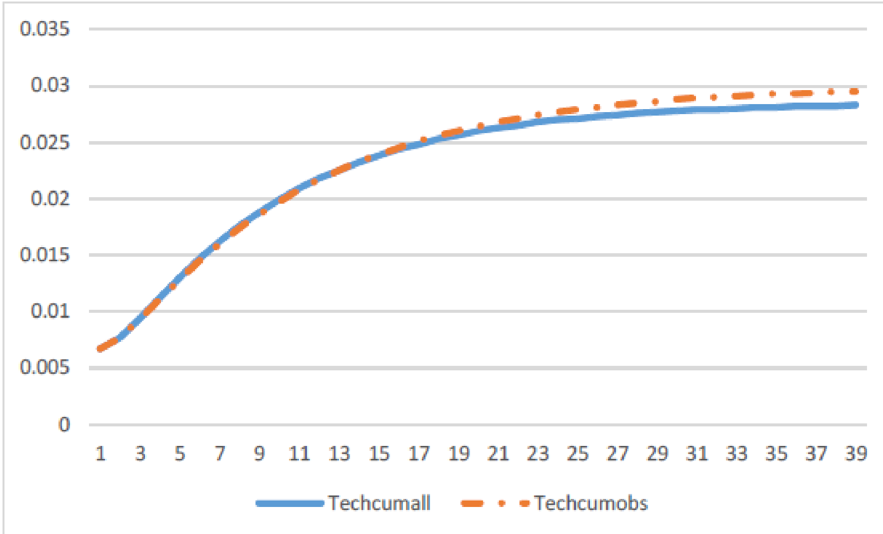

Figure 9: Cumulative Responses of Level of Output to a Technolgy Shock for the An-Schorfheide Model for the Observable-Variable SVAR(2) and DSGE Model

Again, a SSVAR in the observable variables alone, omitting y^S_{t-1}, would be misspecified.

To judge the consequences of the misspecification, just as for the An and Shorfheide (2007) model we compare the relative variances of -.18y^S_{t-1} and $(-1.56\varepsilon^h_t + .96\varepsilon^a_t)$ in Equation (21). The ratio is .33, rather than 9.6 for the An and Schorfheide model, and therefore we would expect a larger bias in the estimated dynamic coefficients in the Poskitt-Yao model.

We can further analyse this bias. Regressing y^S_t upon $\{h_{t-j}, \Delta y_{t-j}\}^M_{j=1}$ in the Poskitt and Yao case we find an R^2 of .7 when $M = 2$. A higher number of lags, namely $M = 50$, is needed in order to be able to reconstruct the latent variable from the observable variables. As there is a high correlation between h_{t-1} and y^S_{t-1}, there will be a bias in the estimates of the coefficient on h_{t-1}. Omitting y^S_{t-1} from Equation (21) yields an estimate of .097 (compared with -0.01), while for Equation (22) it produces .83. Since it is the magnitudes of the estimated coefficients that matter for the impulse responses, the impact of shocks on Δy_t will die out very quickly, even when there is a bias in the estimates, whereas for h_t the difference in the parameter values in Equation (22) for h_{t-1} (.93 and .83) results in a substantial difference in the persistence. The impulse responses are shown in Figures 10 and 11.[16]

Figure 10 is much the same as in Poskitt and Yao (2017). The difference between the responses is due to bias in the h_{t-1} coefficient estimate.[17] However, Figure 11 is very different to what Poskitt and Yao (2017) report. The reason seems to be a computational problem in their graphical output. In their graph there is a big difference between the impulse responses when the latent variable is present and when it is absent, whereas Figure 11 shows that there is almost no difference.

[16]When we fit a VAR(2) in Δy_t and h_t the equation for h_t has coefficients of .94 and -.1 and so for the first few impulse responses there is little difference between the two sets of impulse responses.

[17]To see this, note that after 20 periods ahead we find that the ratio of $(.93/.83)^{20}$ is around 12, which agrees with the figure.

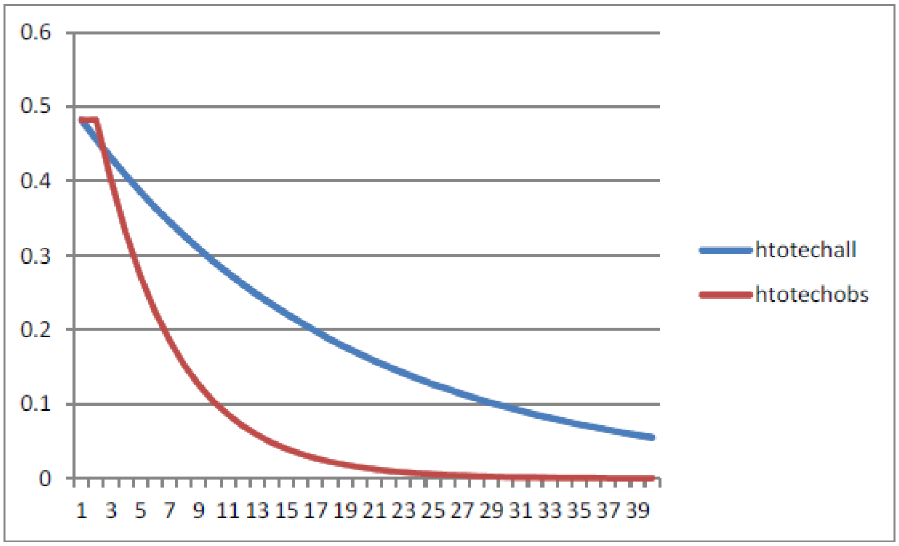

Figure 10: Impulse Response of Hours to a Technology Shock Using the TM of Poskitt and Yao and a VAR(2) in the Observable Variables

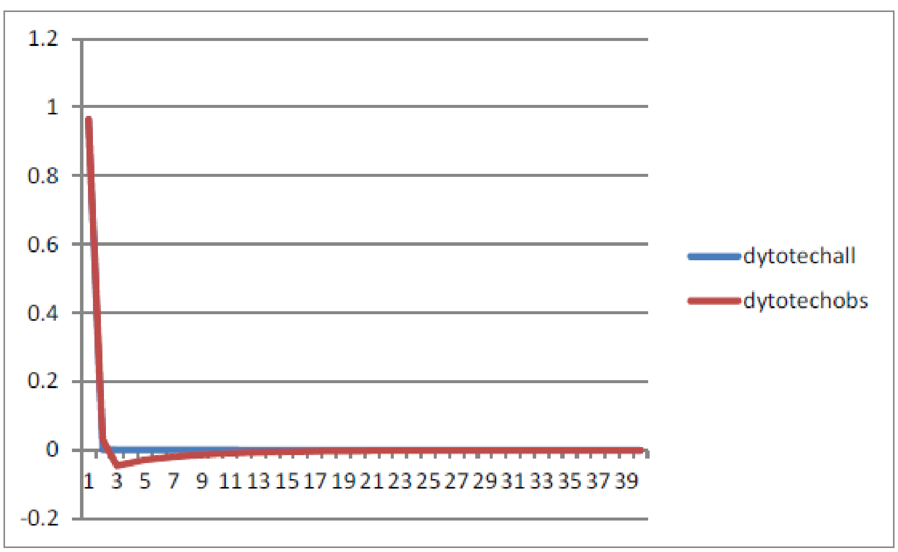

Figure 11: Impulse Response of Growth in Output to a Technology Shock Using the TM of Poskitt and Yao and a VAR(2) in the Observable Variables

3.4 Illustration: The Erceg et. al (2005) Model

Erceg et al. (2005) study the ability of VAR models to match the impulse responses for technology shocks from a DSGE model. They consider two different DSGE models, both containing a permanent technology shock. The first is referred to as the RBC version and in this case they find that the VAR representation exhibits very little truncation error. In the second case nominal rigidities are introduced, amongst other changes, and in this case truncation error occurs.

A four variable VAR is fitted to data simulated from the RBC model and Erceg et al. find a good approximation to the DSGE model impulse responses to a technology shock, which is their main interest. There are four structural shocks and four observable variables - labour productivity lp_t, the log of the consumption to income ratio cy_t, the log of the investment to income ratio iy_t, and hours worked h_t. The model has a unit root in technology so lp_t is stationized, yielding $lp_t^S \equiv lp_t - a_t$. The VAR implied by the DSGE model has the exact form[18]

$$\begin{aligned} lp_t^S &= 1.045lp_{t-1}^S - .04cy_{t-1} + .034iy_{t-1} - .011h_{t-1} && (23) \\ cy_t &= .08lp_{t-1}^S - .926cy_{t-1} + .039iy_{t-1} - .02h_{t-1} && (24) \\ iy_t &= -.22lp_{t-1}^S - .143cy_{t-1} + .875iy_{t-1} - .05h_{t-1} && (25) \\ h_t &= -.173lp_{t-1}^S + .116cy_{t-1} - .075iy_{t-1} + 1.033h_{t-1}. && (26) \end{aligned}$$

So a SSVAR(1) in lp_t^S, cy_t, iy_t and h_t fits exactly. Stationized labour productivity, however, is not observable; the observable-variables SSVAR is in terms of $\Delta lp_t, cy_t, iy_t$ and h_t. Since the error-correction terms appear to be small we would expect that a SVAR(2) in the observable variables could capture the impulse responses quite well, and any differences would only show up in the impulse responses at long horizons. Once again the capital stock (or services from capital) can be substituted out from the production function, and therefore it does not appear in the core endogenous variables. Figures 12 and 13 demonstrates that a SSVAR(2) in observable variables does indeed make a match for moderate horizons.[19]

The second model that Erceg et al. (2005) study has nominal rigidities, but also modifies the production function. In particular, capital services now are the product of a utilization rate and the capital stock k_{t-1}^S. This means that k_{t-1}^S will be a latent core variable, whereas in the first model it could be eliminated. Thus we have two potential sources of problems which may prevent the VAR from giving accurate estimates of the impulse responses. The first is that lp_t^S is not observable, while the second is that the capital stock is also latent. Furthermore, another problem is that there is an additional shock - monetary policy - so there are now five shocks in the model but only the same four observable variables as were used for the RBC model. As the new shock is for monetary policy one might have expected the interest rate r_t to have been added to the VAR as an observable variable. The reason this matters can be seen by comparing the labour productivity VAR equation for this model, Equation (27), to its equivalent for the RBC variant, Equation (23)

[18] The shocks have been omitted from the identities for simplicity.

[19] Because we only work with four decimal places, and the impulse responses are small, this makes for the jagged responses that are seen in the graph.

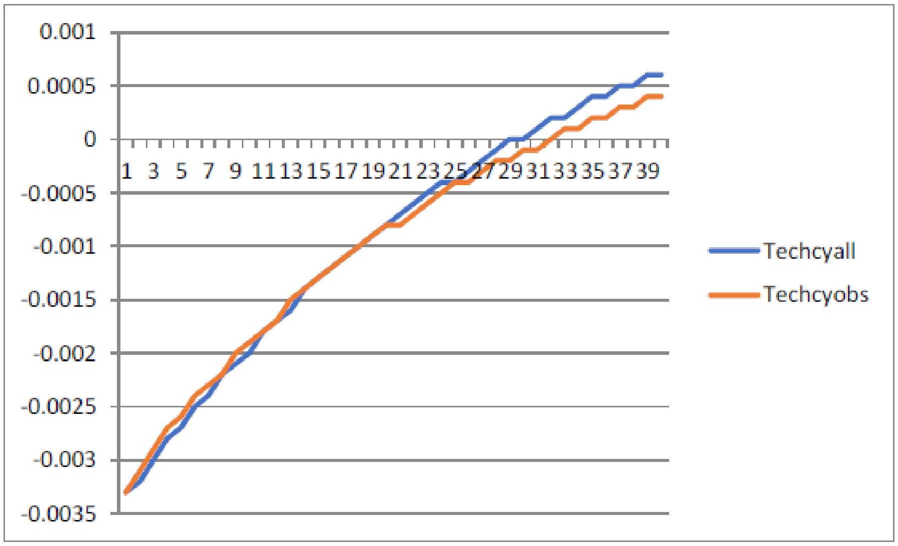

Figure 12: Cumulated Response of Log of Output to a Technology Shock for SVARs Fitted to the Erceg et al RBC Model with All and with just Observed Variables

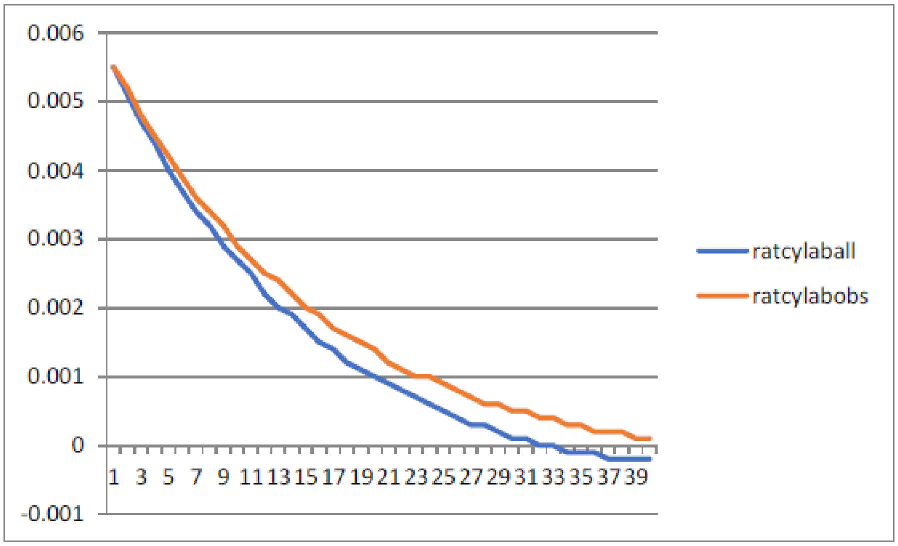

Figure 13: Response of Log of Consumption/Output Ratio to a Labour Supply Shock for VARs Fitted to the Erceg et al RBC Model with All and with just Observed Variables

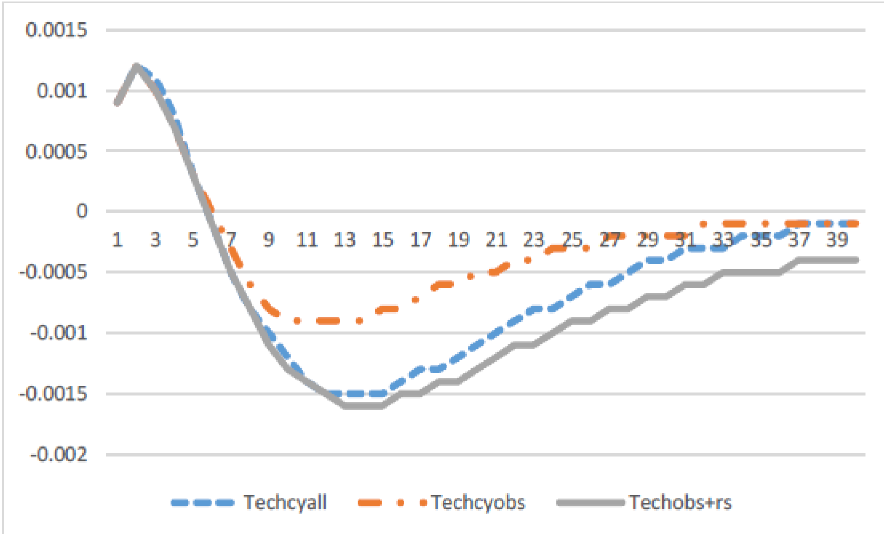

Figure 14: Response of Log of Consumption Output Ratio to a Domestic Technology Shock for SVARs Fitted to Erceg et al Model Output with All, The Erceg et al Observable Variables, and the latter Extended to Include the Interest Rate

$$\begin{aligned} lp_t^S &= 1.16lp_{t-1}^S - .34lp_{t-2}^S + 1.17cy_{t-1} + .12cy_{t-2} - .05iy_{t-1} \\ &\quad + .05iy_{t-2} - .2h_{t-1} + .17h_{t-2} + .0009k_{t-1}^S - .36r_{t-1}. \end{aligned} \tag{27}$$

The key difference is that in Equation (27) r_{t-1} cannot be substituted out and, since it is unobserved, there is no longer a finite-order VAR representation in terms of the stationized variable lp_t^S, even if it were observed. This is what Erceg et al. (2005) find. As before there is also the specification error that comes from replacing lp_t^S with Δlp_t in the VAR. Figure 14 shows the impact of technology shocks upon the log of the consumption output ratio for the DSGE model as well as for an SSVAR(2) fitted with and without the interest rate being treated as an observed variable. It is apparent that much of the truncation error in the long horizon comes from the interest rate being unobserved - for the first three years there is little truncation error.

4 Estimating the Contemporaneous Impulse Responses

4.1 Analysis

We now turn to the second element of making a match between the models, namely the contemporaneous responses. More precisely, we analyse how closely C_0^{SVAR} relates to C_0^{TM}. This is a key element in determining the levels of the impulse responses and it is here that structural information needs to be used. Put another way, it is an SVAR that has to be estimated now rather than just a VAR. In the previous section we set the contemporaneous responses of the shocks equal to that for the TM. If they are not equal, then the responses would originate at different points, even though they would retain the same shape (as that comes from the dynamics, namely the B_j in Equation 3).

To isolate the parameter estimates of the TM which will influence the contemporaneous responses, recall that TMs have the general form of Equation 1, which has Equation 2 as its solution. The latter is an incomplete VAR as the process for the shocks, u_t, has not been specified. For simplicity, let us assume that the shocks are equal to their innovations, ϵ_t. The complete VAR then is

$$z_t = B_1 z_{t-1} + C_0 \epsilon_t,$$

where $B_1 = D$ and $C_0 = G$. So the contemporaneous impulse responses are G. Furthermore, from the complete VAR (the reduced form) the expectations in the TM can be computed, namely $\xi_t = E_t(z_{t+1}) = B_1 z_t$. B_1 can be obtained from the reduced-form VAR. Thus the original representation of the TM in (1) can be re-expressed as

$$A_0 z_{1t} = A_1 z_{t-1} + C\xi_t + \epsilon_t. \tag{28}$$

This expression treats the expectations as an extra endogenous variable. It is apparent from this formulation that the matrix which will influence the contemporaneous responses is $A_0 + C$. This matrix, together with the B_1 from the reduced form, determine G in the solution to the TM.

Assessing the ability to make a match with a SVAR to the contemporaneous impulse responses in the TM is essentially asking whether it is possible to estimate $A_0 + C$. To do this it will often be necessary to adopt *assumptions used in the TM* that are generally not found in SVAR work. In what follows we focus on whether a match can be made given an infinite sample of observations, thereby abstracting from problems arising from the small size of the samples typical in macroeconomic applications. Doing so helps an understanding of what the limits are for a SVAR approach to recover the structural shocks. A recent small New-Keynesian model from the literature is used to illustrate the issues, namely the external sector of the multi-sector model (MSM) of Rees et al. (2016).

Basically the question that needs to be answered is whether there are *enough good instruments* for estimating the parameters in the structural equations of the SVAR (akin to Equation 28) which define the shocks. The SVAR will need to use information from the TM to get these instruments and these will involve three types of restrictions.

1. Exclusion restrictions coming from the structural relations of the TM.
2. Uncorrelated shock assumptions made in the TM.
3. Common factors among the dynamics of the TM.

4.2 Illustration: The External Sector of Rees et.al.'s Multi-Sector Model (MSM)

This is a small New-Keynesian (NK) model of the form

$$y_t = E_t(y_{t+1}) - (r_t - E_t(\pi_{t+1})) + u_{yt} \tag{29}$$
$$\pi_t = \beta E_t(\pi_{t+1}) + \kappa y_t + u_{\pi t} \tag{30}$$
$$r_t = \rho_r r_{t-1} + (1-\rho_r)(\gamma_y y_t + \gamma_\pi \pi_t) + \delta \Delta y_t + \varepsilon_{rt}, \tag{31}$$

where u_{yt} and $u_{\pi t}$ follow AR(1) processes

$$\begin{aligned} u_{yt} &= \rho_y u_{yt-1} + \varepsilon_{yt} \\ u_{\pi t} &= \rho_\pi u_{\pi t-1} + \varepsilon_{\pi t}, \end{aligned}$$

with $\varepsilon_{yt}, \varepsilon_{r\pi}$ and ε_{rt} being white noise processes that are uncorrelated with each other. We assume that π_t and r_t are observed. y_t is actually a stationized variable, however, we will initially abstract from that complication and think of y_t as being an $I(0)$ observable output gap, so as to initially focus on the problems of estimating the SVAR derived C_0 and matching that of the TM. Later we return to the case where the output gap is actually $y_t^S = y_t - a_t$, where a_t is the log of technology which follows an I(1) process.[20]

The NK model in (29)-(31) solves to gives a VAR(1) in y_t, π_t and r_t . Using the parameter values in Rees et al., the SSVAR equation for π_t is

$$\pi_t = .108 y_{t-1} + .269 \pi_{t-1} + .123 r_{t-1} + .0006 \varepsilon_{yt} + .013 \varepsilon_{\pi t} - .009 \varepsilon_{rt}.$$

As discussed above, this provides an expression for expected inflation, namely $\xi_{\pi t} \equiv E_t \pi_{t+1} = .108 y_t + .269 \pi_t + .123 r_t$. In large samples we can recover the expectation by regressing π_{t+1} on y_t, π_t and r_t.[21]

4.2.1 Using COMFAC Restrictions from the TM in the SVAR

Consider the estimation of the New-Keynesian Phillips Curve, Equation (30). It can be converted to an equation (which has innovations as shocks) by multiplying through by a polynomial in the lag operator (1-$\rho_\pi L$) to give

$$(1-\rho_\pi L)\pi_t = \beta(1-\rho_\pi L)\xi_{\pi t} + \kappa(1-\rho_\pi L) y_t + \varepsilon_{\pi,t}. \tag{32}$$

It is evident that the coefficients on π_{t-1}, y_{t-1} and $\xi_{\pi t-1}$ all involve the same parameter ρ_π. Consequently, there is a *common factor* $(1-\rho_\pi L)$ in the three separate lag polynomials. This COMFAC structure was investigated by Hendry and Mizon (1978).

[20] We are also avoiding the problem that in many TMs there are unobservable variables so they would be left out of any structural equation specified by an SVAR, and this is likely to make it harder to make a match. Because it so context dependent we assume that all variables are observable until later when we deal with I(1) variables, where there is only partial observability, but of a from that is not context dependent.

[21] With 10,000 simulated observations from the MSM we obtain .26, .11 and .12 instead of .27, .12 and .11. Even with 200 observations the estimates are accurate; they are .24, .13 and .02.

Now a standard SVAR equation for inflation would be

$$\pi_t = a^0_{21} y_t + a^0_{23} r_t + a^1_{22}\pi_{t-1} + a^1_{21} y_{t-1} + a^1_{23} r_{t-1} + \varepsilon_{\pi t}. \tag{33}$$

Here there are five parameters to be estimated but only three instruments y_{t-1}, r_{t-1} and π_{t-1}. However, when COMFAC restrictions are applied this can be written as

$$\pi_t = a^0_{21} y_t + a^0_{23} r_t + \rho_\pi u_{\pi t-1} + \varepsilon_{\pi t}. \tag{34}$$

Using $\beta = 0.9996$, $\kappa = .036$ and $\rho_\pi = .31$ from Rees et al. (2016) we would find that the SVAR equation implied by MSM would have coefficient values of

$$\begin{aligned} a^0_{21} &= \frac{(.9996 \times .108) + .036}{1 - (.269 \times .9996)1} = .197, \\ a^0_{23} &= \frac{.123}{.731} = .169. \end{aligned}$$

Estimating Equation (34) using simulated values from the MSM (10,000 observations) and instruments π_{t-1}, y_{t-1},r_{t-1} , we would get

$$\pi_t = .192 y_t + .164 r_t + .31 u_{\pi t-1} + \varepsilon_{\pi t},$$

which is an excellent match, enabling the recovery of the MSM shock $\varepsilon_{\pi t}$.

Thinking about this from a conventional SVAR perspective for estimating C_0, Equation (33) could not be estimated, and one action that is often taken is to make the system recursive. This would require at least $a^0_{23} = 0$. Is this reasonable? It is true that r_t is absent from the MSM's Phillips curve, but it should appear in the SVAR equation for inflation because of expectations and it is that which gives $a^0_{23} = .169$. So the COMFAC assumption in the MSM would need to be used to estimate the parameters.

In a similar vein the output equation in the SVAR implied by the MSM model (and parameter values) has the form

$$y_t = .77\pi_t - 29.4 r_t + .95 u_{yt-1}. \tag{35}$$

The COMFAC restriction deliver enough instruments to estimate this equation, producing

$$y_t = .56\pi_t - 22.33 r_t + .952 u_{yt-1} + \varepsilon_{yt},$$

which is a reasonable match to what the MSM model predicts this equation would be, as seen in Equation (35).

The COMFAC restriction used in the MSM model (and TMs more generally) does not come from the microeconomic foundations of the model. It is simply a statistical assumption about the nature of shocks, in particular their autocorrelation, so it could be equally used in an SVAR framework.

4.2.2 Other Restrictions from the TM

Because we can calculate $\xi_{\pi t}$ in (32) from a VAR we might instead formulate the SVAR equation as

$$\pi_t - .9996\xi_{\pi t} = \kappa y_t + \rho_\pi u_{\pi,t-1} + \varepsilon_{\pi,t},$$

since $\beta = .9996$ is prescribed by Rees et al. (2016), rather than being estimated - a very common approach in DSGE models. If that is done then there are two parameters to estimate and three instruments and we would get

$$\pi_t - .9996\xi_{\pi t} = .032y_t + .31u_{\pi,t-1} + \varepsilon_{\pi,t}.$$

This is a good match to the true values. We could add one extra regressor in to the SVAR equation and the answers do not change much. Hence the shock $\varepsilon_{\pi,t}$ can be estimated and impulse response functions found.[22] Thus we don't need to set the parameters on all three lags to zero.

Turning to the final equation in the MSM external system the SVAR equation for r_t implied by the MSM is

$$r_t = .928r_{t-1} + .154y_t - .139y_{t-1} + .107\pi_t.$$

Now this equation could be estimated using the three available instruments y_{t-1}, r_{t-1} and π_{t-1}. But, even with 10,000 observations, the point estimates are very bad, because r_{t-1} is not a good instrument. There are no COMFAC restrictions for this equation as monetary shocks are white noise. However, SVARs do use the assumption of TMs that the structural shocks are uncorrelated with one another, meaning that $\varepsilon_{\pi t}$ and ε_{yt} can be used as extra instruments, if they can be constructed. To do so apply the restrictions discussed above (COMFAC plus parametric) to the SVAR inflation and output equations and compute the residuals $\hat{\varepsilon}_{\pi t}$ and $\hat{\varepsilon}_{yt}$. Using these as extra instruments produces

$$r_t = .928r_{t-1} + .150y_t - .135y_{t-1} + .105\pi_t,$$

which agrees very closely with that implied by the MSM structural equation. When only 200 observations are used the parameter estimates become

$$r_t = .942r_{t-1} + .14y_t - .116y_{t-1} + .078\pi_t,$$

which still represents a good match. Again we could allow for π_{t-1} to appear with little effect.

4.2.3 Issues with I(1) Variables in TMs and SVARs

When there are $I(1)$ variables in the TM then there must be permanent, as well as transitory, shocks. Generally, the log level technology is a permanent shock. This mixture of shocks poses more substantial difficulties in making a SVAR match to the C_0 from the TM.

[22] Once a structural shock such as $\varepsilon_{\pi,t}$ is found impulse responses of the $k'th$ variable to it can be found by regressing the VAR residuals $\hat{e}_{kt}$ against $\hat{\varepsilon}_{\pi,t}$. This is because it is uncorrelated with the other shocks and thus these can be omitted from the regression.

One method for dealing with a mixture of shocks is that due to Shapiro and Watson (1988). They showed that one could separate permanent and transitory shocks by working with some modified equations in the SVAR. As an example, we look at the An and Schorfheide (2007) model, where there is a permanent shock on output from technology while the other shocks two shocks have transitory effects. The following equation for output growth is the Shapiro-Watson formulation and is the SSVAR identity

$$\Delta y_t = -.05y^S_{t-1} + .17\Delta r_t + 1.58\Delta\pi_t + \xi_t, \tag{36}$$

where ξ_t =$.0006\epsilon_{at} - .0002\epsilon_{rt} + .006\epsilon_{gt}$. ξ_t will be a permanent shock.[23] The equation can be estimated using r_{t-1}, π_{t-1} and y^S_{t-1} as instruments. This produces

$$\Delta y_t = -.05y^S_{t-1} + .15\Delta r_t + 1.57\Delta\pi_t + \xi_t, \tag{37}$$

which is a very good match. So the permanent shock ξ_t can be estimated.

Now we need to have all the SVAR equation shocks being uncorrelated in order to be able to compute impulse responses to ξ_t. We therefore look at the inflation equation. It obeys the identity

$$\pi_t = 1.05y^S_t - 1.0y^S_{t-1} - .117r_t + 1.18\pi_{t-1} + 0.0001\varepsilon_{rt} - .0063\varepsilon_{gt}$$

From this, it is immediately apparent that ξ_t will be correlated with the shock in the inflation equation of the SVAR. When the latter equation is estimated (omitting ε_{rt} and ε_{gt}) with $\hat{\xi}_t, \pi_{t-1}, y^S_{t-1}$ and r_{t-1} as instruments we get

$$\pi_t = .065y^S_t - .063y^S_{t-1} - .13r_t + .61\pi_{t-1},$$

which is a very poor match. This arises because of the correlation of the innovations for the Δy_t and π_t SVAR equations. To see that effect suppose we used ε^a_t as the instrument in place of $\hat{\xi}_t$. Then the estimated equation would be

$$\pi_t = -.05y^S_t - 1.0y^S_{t-1} - .12r_t + 1.17\pi_{t-1},$$

and there is now no bias. To separate these shocks one would need to estimate the system, not just this equation. In work such as Gali (1999) impulse responses ξ_t are found by this method. But while ξ_t is a permanent shock it is not equal to the technology shock ϵ_{at}. So it should not be surprising that the impulse responses to ε^a_t from a DSGE model may be different to that for ξ_t from an SVAR.

Now the difficulties with I(1) variables are more complex than above, since we do not observe y^S_{t-1}, only Δy_{t-1}. Re-estimating Equation (37) with y^S_{t-1} replaced by Δy_{t-1} gives

$$\Delta y_t = -.016\Delta y_{t-1} + .20\Delta r_t + 1.71\Delta\pi_t.$$

In this instance the consequence for the estimates of not knowing y^S_{t-1} does not appear to be great.

To check this more carefully let us look at the SVAR for the Phillips curve in the MSM above. There are now two issues. First, $E_t(\pi_{t+1})$ involves the latent variable y^S_t and observed

[23]Pagan and Pesaran (2008) extended this to handle co-integrated variables.

expectations, ξ_t^o, generated from a VAR(2) in the observable variables alone - excludes this. Nevertheless, ξ_t^o and $E_t(\pi_{t+1})$ have a correlation of .91. Second, we need to replace y_t^S with Δy_t. To assess the implications of the second factor, we constrain β to its true value, and estimate using the observable variables as instruments, namely $(\pi_{t-1} - .9996\xi_{t-1}^{MSM})$, $\{\Delta y_{t-j}$, π_{t-j}, $r_{t-j}\}_{j=1}^2$. This produces

$$\pi_t - .9996\xi_t^{MSM} = -0.0005\Delta y_t + 0.28u_{\pi,t-1} + \epsilon_{\pi,t}. \tag{38}$$

So, even if expectations are correctly formed, there will be a bias. If we used the incorrect expectations, based on the observed variables, we would obtain

$$\pi_t - .9996\xi_t^o = -0.0006\Delta y_t + 0.28u_{\pi,t-1} + \epsilon_{\pi,t}. \tag{39}$$

These estimates are very similar to Equation (38), demonstrating the relative importance replacing y_t^S with Δy_t. Finally, using y_t^S we obtain

$$\pi_t - .9996\xi_t^o = -0.01y_t^S + 0.26u_{\pi,t-1} + \epsilon_{\pi,t}. \tag{40}$$

4.3 Implications

The analysis and illustration shows the difficulties that a SVAR can experience in capturing the C_0 from a DSGE model often stem from the fact that the traditional estimation of SVAR models seeks to avoid imposing statistical restrictions (such as COMFAC), exclusion restrictions (in the interest rate equation π_{t-1} does not appear), and other constraints where coefficients are prescribed (for example on $E_t(y_{t+1})$ in the output equation).

In many ways SVARs are about assembling information concerning the dynamics and contemporaneous interactions between variables in the macroeconomy in such a way which, while identified, impose less structure than is included in DSGE models. Traditionally this flexibility has been achieved by using exactly-identified SVARs rather than the over-identified structural equations of the DSGE approach. There are, of course, common restrictions between the two approaches, such as the assumption that the structural shocks are uncorrelated.[24] Alternatively, a criticism of the SVAR approach may be that some of the restrictions frequently used, such as making the system recursive, are driven by convenience, rather than being based on good institutional or economic information.

5 Conclusion

This paper has examined what influences the ability of a SVAR to make a match with the impulse responses of economy-wide models, such as DSGE models. As the latter typically include variables that are not observed in estimation, we have looked at when these unobserved variables can be expressed as a function of the observable variables. Provided the weights on the higher-order lags of the observable variables are low it may be possible to approximate the responses well with a finite-order VAR, which was demonstrated by a simple RBC model.

[24] Note, however, that as DSGE models are often over-identified the estimated structural shocks will not necessarily be uncorrelated - see Liu et al 2018.

Stock variables, which are typically included as latent variables in DSGE models but are not included in SVAR models, emerged as a potentially important factor influence the extent of truncation bias. A small-open economy model showed that the omission of the stock of foreign debt from a VAR could result in truncation biases for real shocks, although even then this was an issue only at longer horizon responses.

DSGE models today often include permanent shocks, such as technology. It was found that in these models truncation biases potentially came from a misspecification. The actual generating process is a latent-factor VECM and not a VAR in terms of observed quantities such as the growth in I(1) variables. Such a model often is relatively easy to estimate and can be implemented so as to safeguard against truncation bias, although in the models studied even a relatively low-order SVAR could overcome this. Nevertheless, such a strategy does not overcome the potential problems that arise from the omission of the stock variables.

Analysis of the well-known Smets Wouters (2007) model found that the major source of the truncation problem was the assumption that the interest rate rule depended on the flex-price output gap and related terms. While theoretically appealing, frequently in applied work log-linearized output and output growth are used instead. Modifying the interest rate in that way mitigated much of the truncation bias. In all, it appears that the extent of truncation bias, while model dependent, often is not substantial and can lessened by strategies such as working with a latent-factor VECM and careful choice of the observed variables.

The second dimension to making a match is how well the SVAR can capture the initial impulse responses rather than the dynamics. It was shown some of the identification in DSGE models is from COMFAC restrictions which are statistical, rather than economic, in nature. Using these, together with other information from the DSGE model, such as exclusion restrictions, may well make a close match between the contemporaneous impulse responses possible. Such restrictions, however, are not commonly adopted in the SVAR literature. Adopting the strong identifying assumptions included in DSGE models would take away a key attractive feature of the SVAR approach, which is the use of as few assumptions as possible in order to produce data-based evidence on the behaviour of the macroeconomy.

References

An, S. and F. Schorfheide (2007), "Bayesian Analysis of DSGE Models," *Econometric Reviews*, 26, 113-172.

Binder, M. and M. H. Pesaran (1995), "Multivariate Rational Expectations Models and Macroeconomic Modelling: A Review and Some New Results" in M.H. Pesaran and M Wickens (eds) *Handbook of Applied Econometrics: Macroeconomics,* Basil Blackwell, Oxford.

Blanchard, O. J. and Kahn, C. M. (1980), "The Solution of Linear Difference Models under Rational Expectations", *Econometrica*, 48, 5, 1305-1311.

Brayton, F. T. Laubach and D. Reifschneider (2014), "The FRB/US Model: A Tool for Macroeconomic Analysis", *FEDS Notes*.

Chari, V. V., P.J. Kehoe and E.R. McGrattan (2005), "A Critique of Structural VARs Using Business Cycle Theory", *Federal Reserve Bank of Minneapolis Working Paper*, 631.

Christiano, L.J., M. Eichenbaum and Robert Vigfusson (2007), "Assessing Structural VARs," in D. Acemoglu, K. Rogoff and M. Woodford (eds) *NBER Macroeconomics Annual*

2006, 21, 1-106.

Christiano, L. J., M. Eichenbaum and C. L. Evans (2005), "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy", *Journal of Political Economy*, 113,1, 1-45.

Del Negro, Schorfhiede, Smets and Wouters (2007), "On the Fit of New Keynesian Models", *Journal of Business and Economic Statistics*, 25, 2, 123-143.

Erceg, C. J., L. Guerrieri and C. Gust (2005), "Can Long-Run Restrictions Identify Technology Shocks?", *Journal of the European Economic Association*, 3, 1237-1278.

Fernández-Villaverde, J., J. F. Rubio-Ramírez, T. J. Sargent and M. W. Watson (2007), "ABCs (and Ds) of Understanding VARs," *American Economic Review*, 97, 1021-1026.

Franchi, M. and A. Vidotto (2013), "A Check for Finite Order VAR Representations of DSGE Models," *Economics Letters*, Elsevier, vol. 120(1), pages 100-103.

Fisher, L. A., H. Huh and A. R. Pagan (2016), "Econometric Methods for Modelling Systems With a Mixture of I(1) and I(0) Variables", *Journal of Applied Econometrics*, 31, 5, 892-911.

Gali. J. (1999), "Technology, Employment, and the Business Cycle: Do Technology Shocks Explain Aggregate Fluctuations?", *The American Economic Review*, 89, 249-271.

Giacomini, R. (2013), "The Relationship Between VAR and DSGE Models", in T. B. Fomby, L. Kilian and A. Murphy (eds) *VAR Models in Macroeconomics – New Developments and Applications: Essays in Honor of Christopher A. Sims*, *Advances in Econometrics*, 32, 1-25.

Hendry, D. F. and G. Mizon (1978), "Serial Correlation as a Convenient Simplification, Not a Nuisance: A Comment on a Study of the Demand for Money by the Bank of England," *Economic Journal*, 88, 549-63.

Justiniano, A. and B. Preston (2010), "Can Structural Small Open-Economy Models Account for the Influence of Foreign Distrurbances?" *Journal of International Economics*, 81, 61-74.

Kapetanios, G., A.R. Pagan and A. Scott (2007), "Making a Match: Combining Theory and Evidence in Policy-oriented Macroeconomic Modeling", *Journal of Econometrics*, 136, 505-594.

Liu, P. and T. Konstantinos (2012), "DSGE Model Restrictions for Structural VAR Identification," *International Journal of Central Banking*, 8, 61-95.

Liu, X., A. R. Pagan and T. Robinson (2018), "Critically Assessing Estimated DSGE Models: A Case Study of a Multi-sector Model", *Centre for Applied Macroeconomic Analysis Working Paper*, Australian National University, 2018-04.

Morris, S.D. (2016), "VARMA Representation of DSGE Models", *Economics Letters*, 138, 30-33.

Morris, S.D. (2017), "DSGE Pileups", *Journal of Economic Dynamics and Control*, 74, 56-86.

Pagan, A. R. (2017) "Some Consequences of Using 'Measurement Error Shocks' When Estimating Time Series Models", *Centre for Applied Macroeconomic Analysis Working Paper*, Australian National University, 2017-12.

Pagan, A. R. and M. H. Pesaran (2008) "Econometric Analysis of Structural Systems with Permanent and Transitory Shocks", *Journal of Economic Dynamics and Control*, 32, 10, 3376-3395.

Poskitt, D. and Yao, W. (2017), “Vector Autoregressions and Macroeconomic Modelling: an Error Taxonomy”, *Journal of Business and Economics Statistics*, 35, 3, 407-419 .

Ravenna, F. (2007) “Vector Autoregressions and Reduced Form Representations of DSGE Models”, *Journal of Monetary Economics*, 54, 7, 2048-2064.

Rees, D.M., P. Smith and J. Hall (2016), “A Multi-sector Model of the Australian Economy”, *Economic Record*, 92, 298, 374-408.

Schmitt-Grohe, S and M. Uribe (2003). “Closing Small Open Economy Models,” *Journal of International Economics*, 61, 163-185.

Shapiro M and Watson M. (1988), “Sources of Business Cycle Fluctuations”, in *NBER Macroeconomics Annual*, 3, 111-148.

Smets, F. and R. Wouters (2007), “Shocks and Frictions in US Business Cycles: a Bayesian DSGE Approach”, *American Economic Review*, 97, 586-606.

Uhlig, F. H. (1999), “A Toolkit for Analysing Nonlinear Dynamic Stochastic Models Easily”, in R. Marimon and A. Scott (eds), *Computational Methods for the Study of Dynamic Economies*, Oxford University, Press, 30-61.